Remote Learning Packet

NB: Please keep all work produced this week. Details regarding how to turn in this work will be forthcoming.

March 30 - April 3, 2020

Course: 11 Calculus I

Teacher: Mr. Simmons michael.simmons@greatheartsirving.org

Weekly Plan:

Monday, March 30

☐ Worksheet 4.2.I

Tuesday, March 31

☐ Worksheet 4.2.I answer key

Wednesday, April 1

☐ Worksheet 4.2.II

Thursday, April 2

☐ Worksheet 4.2.II answer key

Friday, April 3

☐ Worksheet 4.3.I

Statement of Academic Honesty

I affirm that the work completed from the packet is mine and that I completed it independently.

____

Student Signature

I affirm that, to the best of my knowledge, my child completed this work independently

____

Parent Signature

Monday, March 30

Dearest students, hello! I hope you've enjoyed your leisure over the past two weeks. While I am heartbroken that I can't be with you in person to guide you through the wonderful beauties of mathematics over these next few weeks, I see some good coming from the circumstances, as each of you will be challenged - and surely will rise to meet the challenge - to engage in mathematical inquiry in a more independent way. The frustrations you will feel in your isolated struggle for mathematical understanding, no matter your success, will doubtless sow seeds of true mathematical thought - and winter always ends in a fruitful spring. The essentially collaborative nature of our class has unfortunately (though necessarily) been temporarily suspended, so take courage as you embark on this lonely adventure, and look forward to our reunion in good time. Have fortitude, and I look forward to seeing you all again.

In the meantime, your individual adventures in math will be guided by written instructions contained in packets like this one. Your first instruction is to review the new vocabulary list. Make sure to go over every new definition in detail. Then, complete worksheet 4.2.I. As you do so, recall how much time we spend graphing derivatives and second derivatives, and then discussing the relationships between those graphs and the original. You'll probably want to use that knowledge on this worksheet.

Tuesday, March 31

Carefully read through the answer key to worksheet 4.2.I.

Wednesday, April 1

Complete worksheet 4.2.II.

Thursday, April 2

Carefully read through the answer key to worksheet 4.2.II.

Friday, April 3

Remember how last year you sketched graphs of polynomials? But there was always the frustration that while we knew a local maximum or local minimum lay between two turning points, we were never sure exactly where the hill or valley peaked or troughed. Well. Now we have calculus! Worksheet 4.3.I let's you show off the precision with which the powerful tools of calculus let you visualize polynomial functions. Before getting started on that worksheet, make sure to review concavity and inflection points in the vocabulary list. Then complete worksheet 4.3.I.

Vocabulary List

Calculus I
Mr. Simmons
March 30, 2020

Definition (RELATION)**.** A relation is a set of ordered pairs consisting of elements from two sets. A relation relates input values, the first components of the ordered pairs, to output values, the second components of the ordered pairs. A relation is represented by a table, an equation, or a graph.

Definition (FUNCTION)**.** A function is a relation that has exactly one output for every input. A function takes an input value and outputs a function value.

Definition (DOMAIN)**.** A domain is the set of all input values for which a function is defined.

Definition (RANGE)**.** A range is the set of output values that result from all the input values of a function.

Definition (LIMIT – INFORMAL)**.** A limit is the value that a function value approaches as the input approaches some value.

Definition (LIMIT – FORMAL)**.** To say that L is the limit of a function value $f(x)$ as the input x approaches c is to say that we can make $f(x)$ arbitrarily[1] close to L by taking x sufficiently[2] close to c.

Definition (LIMIT – ε-δ)**.** The limit of a function value $f(x)$ as x approaches c is the number L if and only if, given any positive ε, there exists a positive δ such that, for all x strictly within distance δ of c (except c itself), $f(x)$ is strictly within distance ε of L.

Definition (CONTINUITY AT A POINT)**.** A function is continuous at a point of its domain if and only if its limit equals its function value at that point.

Definition (CONTINUITY OF A FUNCTION)**.** A function is continuous if and only if it is continuous at each point of its domain.

Definition (DERIVATIVE)**.** The derivative of a function at a chosen input value, when it exists, is the slope of the tangent line of the function's graph at that point.

[1] **arbitrary (adj.)** c. 1400, "deciding by one's own discretion, depending on one's judgment," from Latin *arbitrarius* "of arbitration," hence "depending on the will, uncertain," from *arbiter*. The meaning in English gradually descended to "capricious, ungoverned by reason or rule, despotic" (1640s). Related: Arbitrarily; arbitrariness. —Etymonline.com

[2] **sufficient (adj.)** early 14c., from Old French *soficient* "satisfactory," or directly from Latin *sufficientem* "adequate," present participle of *sufficere* "to supply as a substitute," from *sub* "up to" + combining form of *facere* "to make, to do." —Etymonline.com

Definition (RATE OF CHANGE). A rate of change is the change of an output value relative to the change of the input value.

Definition (AVERAGE RATE OF CHANGE). An average rate of change is the change of a function value over an interval divided by the interval's width.

Definition (INSTANTANEOUS RATE OF CHANGE). An instantaneous rate of change is the limit of an average rate of change on an interval as the interval's width approaches zero.[3]

Definition (VELOCITY). Velocity is displacement relative to time.

Definition (AVERAGE VELOCITY). Average velocity is the displacement of an object over a time interval divided by the interval's duration.

Definition (INSTANTANEOUS VELOCITY). Instantaneous velocity is the limit of average velocity over a time interval as the interval's duration approaches zero.[4]

Definition (INCREASING FUNCTION). A function f defined on a domain D is said to be increasing on an open interval I if and only if $f(b) > f(a)$ for any two inputs a and b in I where $b > a$. If f is increasing on every open interval in D, we say simply that f is increasing.

Definition (DECREASING FUNCTION). A function f defined on a domain D is decreasing on an open interval I if and only if $f(b) < f(a)$ for any two input values a and b in I where $b > a$. If f is decreasing on every open interval in D, we say simply that f is decreasing.

Definition (GLOBAL MAXIMUM). A function f defined on a domain D has a global (or absolute) maximum point at c if and only if $f(c) \geq f(x)$ for all x in D. The value of the function at a maximum point is called the maximum value of the function.

Definition (GLOBAL MINIMUM). A function f defined on a domain D has a global (or absolute) minimum point at c if and only if $f(c) \leq f(x)$ for all x in D. The value of the function at a minimum point is called the minimum value of the function.

Definition (LOCAL MAXIMUM). A function f defined on a domain D is said to have a local (or relative) maximum point at the point c if and only if there exists some $\varepsilon > 0$ such that $f(c) \geq f(x)$ for all x in D within distance ε of c.

Definition (LOCAL MINIMUM). A function f defined on a domain D is said to have a local (or relative) minimum point at the point c if and only if there exists some $\varepsilon > 0$ such that $f(c) \leq f(x)$ for all x in D within distance ε of c.

Definition (EXTREMUM). An extremum is a maximum or a minimum. That is, given a function f, $f(c)$ is an extremum of f if and only if $f(c)$ is a maximum or a minimum of f.

Definition (CONCAVE UP). A function f defined on a domain D is said to be concave up on an open interval I if and only if $f'(b) > f'(a)$ for any two inputs a and b in I where $b > a$.

[3] Note that *instantaneous rate of change* and *derivative* are synonyms.

[4] Note that, unlike average velocity, instantaneous velocity is not a scientific concept, but a mathematical one. No instrument can measure change over a time interval of no duration. Note also that, since *instantaneous rate of change* and *derivative* are synonyms, instantaneous velocity is the derivative of displacement.

Definition (CONCAVE DOWN)**.** A function f defined on a domain D is said to be concave down on an open interval I if and only if $f'(b) < f'(a)$ for any two inputs a and b in I where $b > a$.

Definition (CRITICAL POINT)**.** A function f has a critical point at c if and only if $f(c)$ exists and either $f'(c) = 0$ or $f'(c)$ does not exist.

Definition (INFLECTION POINT)**.** A function f has an inflection point at c if and only if $f''(c) = 0$.

Worksheet 4.2.I

Calculus I
Mr. Simmons

1. Find the absolute minimum and maximum values for $f(x) = x^2 - 1$ on the interval $[-1, 2]$.

2. Find the absolute minimum and maximum values for $f(x) = -\sqrt{5 - x^2}$on the interval $\left[-\sqrt{5}, 0\right]$.

3. The first derivative of a function $y = f(x)$ is known to be $f'(x) = (x - 1)(x + 2)$. Use the derivative to determine:

   (a) What are the critical points of f?

   (b) On what intervals is f increasing and decreasing?

   (c) At what points, if any, does f assume local minimum and maximum values?

4. Use calculus to find the absolute maximum and minimum values of the function $f(x) = 4 - x^2$ on the interval $[-3, 1]$.

5. Consider the function $f(x) = 6x - x^2$. Use the function's first derivative to find the intervals on which the function is increasing and decreasing. Then, identify the function's local and extreme values – where they are assumed and what their values are.

Worksheet 4.2.I – Answer Key

Calculus I
Mr. Simmons

1. Find the absolute minimum and maximum values for $f(x) = x^2 - 1$ on the interval $[-1, 2]$.

   Solution. The key to these problems is to recognize the absolute extrema occur either at local extrema or at endpoints of a closed interval. To find local extrema, find where $f'(x)$ is 0:

$$\begin{aligned} f'(x) &= 0 \\ 2x &= 0 \\ x &= 0 \end{aligned}$$

   and then find the value of $f(x)$ there:

$$\begin{aligned} f(0) &= (0)^2 - 1 \\ &= -1. \end{aligned}$$

   Since, therefore, -1 is a local extremum, it has a chance of being an absolute extremum. But let's test the endpoint -1 and 2:

$$\begin{aligned} f(-1) &= (-1)^2 - 1 \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

   and

$$\begin{aligned} f(2) &= (2)^2 - 1 \\ &= 4 - 1 \\ &= 3. \end{aligned}$$

   Of -1, 0, and 3, -1 is the absolute minimum and 3 is the absolute maximum.

2. Find the absolute minimum and maximum values for $f(x) = -\sqrt{5 - x^2}$on the interval $\left[-\sqrt{5}, 0\right]$.

   Solution. Finding where $f'(x) = 0$ will tell us where any local extrema are:

$$\begin{aligned} f'(x) &= 0 \\ \frac{x}{\sqrt{5 - x^2}} &= 0 \\ x &= 0 \end{aligned}$$

(which, note, happens to be an endpoint), and so

$$\begin{aligned} f(0) &= -\sqrt{5-(0)^2} \\ &= -\sqrt{5}. \end{aligned}$$

Testing the other endpoint $-\sqrt{5}$, we get

$$\begin{aligned} f\left(-\sqrt{5}\right) &= -\sqrt{5-\left(-\sqrt{5}\right)^2} \\ &= -\sqrt{5-5} \\ &= 0. \end{aligned}$$

So the absolute minimum is $-\sqrt{5}$ and the absolute maximum is 0.

3. The first derivative of a function $y = f(x)$ is known to be $f'(x) = (x-1)(x+2)$. Use the derivative to determine:

   (a) What are the critical points of f?

   Solution. Remember that a critical point of f is a point where $f(x)$ is defined, but where $f'(x)$ either is zero or doesn't exist. To find critical points, we find where $f'(x) = 0$:

   $$\begin{aligned} f'(x) &= 0 \\ (x-1)(x+2) &= 0 \\ x &= 1, -2. \end{aligned}$$

   The critical points of f are at $x = 1$ and $x = -2$.

   (b) On what intervals is f increasing and decreasing?

   Solution. We know $f(x)$ will be increasing wherever $f'(x)$ is positive, and it will be decreasing wherever $f'(x)$ is negative. Well, by our knowledge of polynomials, we know that $f'(x)$ is positive on $(-\infty, -2)$ and $(1, \infty)$ and negative on $(-2, 1)$, so $f(x)$ is increasing on $(-\infty, -2)$ and $(1, \infty)$ and decreasing on $(-2, 1)$.

   (c) At what points, if any, does f assume local minimum and maximum values?

   Solution. The function f will assume local extrema wherever $f'(x) = 0$, which happens at $x = 1, -2$. Since $x = -2$ is where $f(x)$ goes from increasing to decreasing, f will have a local maximum here, and similarly it will have a local minimum at $x = 1$.

4. Use calculus to find the absolute maximum and minimum values of the function $f(x) = 4 - x^2$ on the interval $[-3, 1]$.

   Solution. First we find local extrema:

   $$\begin{aligned} f'(x) &= 0 \\ -2x &= 0 \\ x &= 0; \\ f(0) &= 4-(0)^2 \\ &= 4. \end{aligned}$$

Then we test endpoints:

$$\begin{aligned} f(-3) &= 4 - (-3)^2 \\ &= 4 - 9 \\ &= -5; \\ f(1) &= 4 - (1)^2 \\ &= 4 - 1 \\ &= 3. \end{aligned}$$

Of 4, -5, and 3, -5 is the absolute minimum (occuring at $x = -3$) and 4 is the absolute maximum (occuring at $x = 0$).

5. Consider the function $f(x) = 6x - x^2$. Use the function's first derivative to find the intervals on which the function is increasing and decreasing. Then, identify the function's local and extreme values – where they are assumed and what their values are.

   Solution. To find where f is increasing and decreasing, we find where $f'(x)$ is positive and negative: $f'(x) = -2x + 6$ is positive on $(-\infty, 3)$ and negative on $(3, \infty)$, so f is increasing on $(-\infty, 3)$ and decreasing on $(3, \infty)$. Since $f'(x) = 0$ at $x = 3$, that's where f has the local extremum $f(3) = 6(3) - (3)^2 = 9$. Since f is changing from increasing to decreasing, 9 is a local maximum, not a minimum.

Worksheet 4.2.II

Calculus I
Mr. Simmons

1. Find the absolute minimum and maximum values for $f(x) = x^{\frac{4}{3}}$ on the interval $[-1, 8]$ and determine where they are assumed.

2. Consider the function $g(t) = -3t^2 + 9t + 5$. Use the function's first derivative to find the intervals on which the function is increasing and decreasing. Then identify the function's local extreme values, if there are any, stating where they are assumed and what their values are.

3. Consider the function $f(x) = (x + 1)^2$, for $-\infty < x \leq 2$. Identify the function's local extreme values in the given domain and determine where they are assumed.

4. Consider the function $y = -x^3 + 6x^2$, for $-1 \leq x \leq 5$. Identify the function's local extreme values in the given domain and determine where they are assumed.

Worksheet 4.2.II – Answer Key

Calculus I
Mr. Simmons

1. Find the absolute minimum and maximum values for $f(x) = x^{\frac{4}{3}}$ on the interval $[-1, 8]$ and determine where they are assumed.

   Solution. We have

$$\begin{aligned} f'(x) &= 0 \\ \frac{4}{3}x^{\frac{1}{3}} &= 0 \\ x^{\frac{1}{3}} &= 0 \\ x &= 0; \\ f(0) &= (0)^{\frac{4}{3}} \\ &= 0; \\ f(-1) &= (-1)^{\frac{4}{3}} \\ &= 1; \\ f(8) &= (8)^{\frac{4}{3}} \\ &= 16; \end{aligned}$$

   so $f(x)$, on the interval $[-1, 8]$, has the absolute minimum 0 at $x = 1$ and the absolute maximum 16 at $x = 8$.

2. Consider the function $g(t) = -3t^2 + 9t + 5$. Use the function's first derivative to find the intervals on which the function is increasing and decreasing. Then identify the function's local extreme values, if there are any, stating where they are assumed and what their values are.

   Solution. We have

$$g'(t) = -6t + 9,$$

   which is positive on $\left(-\infty, \frac{3}{2}\right)$ and negative on $\left(\frac{3}{2}, \infty\right)$ so $g(t)$ is increasing on $\left(-\infty, \frac{3}{2}\right)$ and decreasing on $\left(\frac{3}{2},\infty\right)$. And $g'(t)=0$ when $t=\frac{3}{2}$, so g has a local extremum there of

$$\begin{aligned}
g\left(\frac{3}{2}\right) &= -3\left(\frac{3}{2}\right)^2+9\left(\frac{3}{2}\right)+5\\
&= -3\left(\frac{9}{4}\right)+\frac{27}{2}+5\\
&= -\frac{27}{4}+\frac{27}{2}+5\\
&= -\frac{27}{4}+\frac{54}{4}+\frac{20}{4}\\
&= \frac{47}{4}.
\end{aligned}$$

Since this extremum occurs where $g(t)$ changes from increasing to decreasing, it is a local maximum.

3. Consider the function $f(x)=(x+1)^2$, for $-\infty<x\leq 2$. Identify the function's local extreme values in the given domain and determine where they are assumed.

   Solution. We have

$$\begin{aligned}
f'(x) &= 0\\
2(x+1) &= 0\\
x+1 &= 0\\
x &= -1;\\
f(-1) &= ((-1)+1)^2\\
&= 0;
\end{aligned}$$

   so f has a local extremum of 0 at $x=-1$. There are a few ways to tell that this extremum is a minimum, one of which is to notice that $f(x)$ is a positive quadratic. There is a second local extremum at $x=2$:

$$\begin{aligned}
f(2) &= ((2)+1)^2\\
&= 9.
\end{aligned}$$

   Since $f(x)$ is increasing up until $x=2$, this must be a local maximum.

4. Consider the function $y=-x^3+6x^2$, for $-1\leq x\leq 5$. Identify the function's local extreme values in the given domain and determine where they are assumed.

Solution. We have

$$\begin{aligned}
y' &= 0 \\
-3x^2 + 12x &= 0 \\
x^2 - 4x &= 0 \\
x(x-4) &= 0 \\
x &= 0, 4; \\
y &= -(0)^3 + 6(0)^2, -(4)^3 + 6(4)^2 \\
&= 0, 32;
\end{aligned}$$

so this function has local extrema of 0 and 32 at $x = 0$ and $x = 4$, respectively. Since the function is a negative cubic, 0 must be a local minimum and 32 must be a local maximum. (To clarify this, graph y.) The endpoints -1 and 5 also give us local extrema:

$$\begin{aligned}
f(-1) &= -(-1)^3 + 6(-1)^2 \\
&= 7; \\
f(5) &= -(5)^3 + 6(5)^2 \\
&= 5^2(-5+6) \\
&= 25.
\end{aligned}$$

Since y is decreasing through both $x = -1$ and $x = 5$, both 7 and 25 are local maxima.

Worksheet 4.3.I

Calculus I
Mr. Simmons

1. Using the first and second derivatives, sketch a graph for $y = x^2 - 4x + 10$.
2. Using the first and second derivatives, sketch a graph for $y = 3x^3 - 4x^2 - 2$.
3. Using the first and second derivatives, sketch a graph for $y = x^4 - 4x^3 + 1$.
4. Using the first and second derivatives, sketch a graph for $y = \sin^2(x) + 1$.