GreatHearts Irving

Remote Learning Packet

NB: Please keep all work produced this week. Details regarding how to turn in this work will be forthcoming.

April 13 - 17, 2020
Course: 11 Calculus I
Teacher(s): Mr. Simmons

Weekly Plan:

Monday, April 13
☐ First proof of Theorem 1.

Tuesday, April 14
☐ Review first proof of Theorem 1.
☐ Second proof of Theorem 1.

Wednesday, April 15
☐ Attempt proofs of the rest of the theorems.

Thursday, April 16
☐ Attempt proofs of the rest of the theorems.

Friday, April 17
☐ Attempt proofs of the rest of the theorems.

Statement of Academic Honesty

I affirm that the work completed from the packet is mine and that I completed it independently.

I affirm that, to the best of my knowledge, my child completed this work independently

____
Student Signature

____
Parent Signature

Monday, April 13

Last week, you were instructed to attempt to prove the theorems in the "Important Derivative Theorems" handout (which I have since renamed "Eight Important Derivative Theorems." Remember: math takes time. Be patient. Don't be frustrated if you have to stare at a theorem for a very long time before knowing how even to start proving it. One approach that will help get you started is analyzing every single word in the theorem according to its rigorous definition. (That means you should look back at the vocabulary handout.)

I am including in this packet two proofs of Theorem 1. Please

1. Read through the first proof carefully, making sure to go slow enough that you understand each sentence before moving on. If you need to go back and look up vocabulary, take the time to do that.
2. In the handout, there are instructions for you to prove part of the proof yourself. You may leave that until tomorrow.
3. Use any spare time to review and solidify vocabulary, or, if you would like, to continue trying to prove the rest of the theorems.

Tuesday, April 14

1. As instructed in the "Eight Important Derivative Theorems" handout, fill in the blank page for the first proof of Theorem 1.
2. Use any spare time to review and solidify vocabulary, or, if you would like, to continue trying to prove the rest of the theorems.

Wednesday, April 15

1. Read through the second proof of Theorem 1 (after having read the preceding paragraph). It is not essentially different from the first proof, but it is stated more concisely.
2. Read through the rest of the theorems, and choose exactly one that you think you have the best chance of being able to prove. Spend the remainder of your time proving that one.

Thursday, April 16

1. Out of the theorems that you have not yet finished proving, select the theorem that you think you have the best chance of being able to prove. Spend today's 40 minutes proving it. Turn in your proof with this packet.

Friday, April 17

1. Out of the theorems that you have not yet finished proving, select the theorem that you think you have the best chance of being able to prove. Spend today's 40 minutes proving it. Turn in your proof with this packet.

Eight Important Derivative Theorems

With a proof of Theorem 1

Calculus I
Mr. Simmons

Several important results emerge while studying the relationships between a function's extrema, shape, and its derivatives. Among these results we have the following theorems. Read and work through the theorems and proofs below.

Theorem 1. *If a local extremum occurs at an interior point, $x = c$, and $f'(c)$ exists, then $f'(c) = 0$.*

Proof. The following proof is dangerous: the reader will be tempted to get lost in the abstract manipulation of symbols and forget what the symbols refer to. To guard against this danger, I highly encourage you to sketch out graphs and diagrams to help you understand every step of the proof. Remember that the theorem itself is actually quite simple. All it says is that, at the peak of a hill or the trough a valley, the tangent line will be flat.

Breaking the proof into cases. We start, as always, by supposing the hypotheses: let f be a function with a local extremum at an interior point c of its domain, and suppose $f'(c)$ exists. Remember that what it means for $f(c)$ to be a local extremum of f is that it is either a local maximum or a local minimum of f. Let's deal with these two cases separately. Once we prove the theorem for each of the two cases, since they are the only two possibilities, then we will have proven the theorem completely. This is called a **proof by cases**.

Case where the extremum is a maximum

Consider the case where $f(c)$ is a local maximum of f. We want to show that $f'(c) = 0$, and our plan is to do that indirectly by showing first that $f'(c) \not> 0$ and then that $f'(c) \not< 0$.[1] Since the only number that is neither greater nor less than zero is zero itself, this will be sufficient to show that $f'(c) = 0$.

Since, for each of these two steps, we're trying to prove a negative (i.e., that the derivative is *not* greater than or *not* less than zero), we're going to use the method called **proof by contradiction**. The basic outline of the proof is that we're going to assume $f'(c)$ is positive, meaning that there's a positively sloped tangent line at c, and then find an x just a little bit to the right of c that yields $f(x) > f(c)$, a contradiction since $f(c)$ is a local maximum. Then we'll do the same for $f'(c)$ being negative by choosing an x just a bit to the left of c. Don't move on in the proof until you have a clear picture of the outline of the proof. Draw a diagram if it's helpful.

[1] Note that $a \not> b$ is equivalent to $a \leq b$, by the trichotomy property of the real numbers, which states that for any two real numbers a and b (not necessarily distinct), exactly one of the following is true: (1) $a = b$; (2) $a < b$; or (3) $a > b$. Similarly, $a \not< b$ is equivalent to $a \geq b$.

Proof that the derivative is nonpositive. Suppose for contradiction that $f'(c) > 0$. (That is, suppose for contradiction that the tangent line at c has a positive slope.) What this means, by the definition of a derivative, is that

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} > 0.$$

Let's give this limit a name. Call it m, since it represents the slope of a tangent line. So

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = m,$$

where m is some positive number. This, by the definition of a limit, is equivalent to saying that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that, for all x,

$$0 < |x - c| < \delta \quad \Longrightarrow \quad \frac{f(x) - f(c)}{x - c} - m \; < \varepsilon.$$

(Remember that $|x - c| < \delta$ is the same as saying that x is strictly within δ of c, and, similarly,

$$\frac{f(x) - f(c)}{x - c} - m \; < \varepsilon$$

is equivalent to saying

$$\frac{f(x) - f(c)}{x - c} \text{ is strictly within } \varepsilon \text{ of } m.)$$

Now we just a need to choose an x value "a bit to the right of" c (meaning close enough for the limit definition to apply, i.e., within δ of c) that will yield $f(x) > f(c)$ in such a way as to contradict $f(c)$ being a local maximum. We need to get within δ of c for the limit definition to apply, but how close do we have to get to c for the definition of a local maximum to apply? To help us determine which x to pick, let's look at what it means for $f(c)$ to be a local maximum: it means that there is some open interval (a, b) containing c such that $f(x) \leq f(c)$ for all $x \in (a, b)$.

So we need to fix an x value that is in (a, b), within δ of c, and to the right of c. But wait, what's our δ? The limit definition says that for any positive ε, there exists a positive δ such that etc. So we can choose any ε we want, and it will give us a δ. Okay, let's let ε be any positive number. Therefore, by the definition of a limit, there exists a δ such that for all x,

$$0 < |x - c| < \delta \quad \Longrightarrow \quad \frac{f(x) - f(c)}{x - c} - m \; < \varepsilon, \tag{1}$$

where ε is no longer variable, but fixed. There, we have our fixed δ. Now let's choose an x that's in (a, b), within δ of c, and to the right of c.

To do this we let x be a point that is strictly greater than c but strictly less than both b and $c + \delta$. We can do this because we know both $c < b$ and $c < c + \delta$, so there's room in between c and the lesser (the minimum) of b and $c + \delta$ to select an x value.

Since $x \in (a, b)$, by the definition of a local maximum, therefore $f(x) \leq f(c)$.

Now we just need to show that $f(x) > f(c)$ (where x is now fixed, not variable), which will contradict the fact that $f(c)$ is a local maximum. Well, since x is within δ of c but not c itself, then by implication (1), we know that

$$\frac{f(x) - f(c)}{x - c} - m \; < \varepsilon, \tag{2}$$

where ε is the positive number we fixed earlier. Can we get from here to the statement that $f(x) > f(c)$?

Look at the expression in the absolute value bars. It's either negative or nonnegative, so let's consider those two cases:

Case 1. Suppose the expression in the absolute value bars in (2) is nonnegative, that is, that

$$\frac{f(x) - f(c)}{x - c} - m \geq 0.$$

Then

$$\begin{aligned} \frac{f(x) - f(c)}{x - c} &\geq m \\ f(x) - f(c) &\geq m(x - c) \\ f(x) &\geq f(c) + m(x - c) \\ f(x) &> f(c), \end{aligned}$$

since, remember, m is positive ($m > 0$) and x is to the right of c ($x > c$, and so $x - c > 0$), so we know $m(x - c)$ is strictly positive, being the product of strictly positive values.

Case 2. Suppose the expression is negative. Then inequality (2) becomes

$$\begin{aligned} -\left(\frac{f(x) - f(c)}{x - c} - m\right) &< \varepsilon \\ \frac{f(x) - f(c)}{x - c} - m &> -\varepsilon \\ \frac{f(x) - f(c)}{x - c} &> m - \varepsilon \\ f(x) - f(c) &> (m - \varepsilon)(x - c) \\ f(x) &> f(c) + (m - \varepsilon)(x - c). \end{aligned}$$

We're so close. We know $x - c$ is positive, so if we can just show that $m - \varepsilon$ is positive, or at least nonnegative, then we'll be good—we will have shown that $f(x) > f(c)$. But the problem is, we have no way of showing that $m - \varepsilon$ is nonnegative, that is, that $\varepsilon \leq m$. Our ε was defined as some arbitrary positive number. We simply don't know whether it's greater than, equal to, or less than m.

But wait, we *chose* ε. Let's just go back and, instead of choosing just any arbitrary positive number for ε, let's choose an ε that will make $m - \varepsilon$ nonnegative. Go back to where we said, "let ε be any positive number," and edit it to say, "let $\varepsilon = m$."[2] The proof up until now will not be affected, but this will help us with Case 2. So now our inequality is

$$f(x) > f(c) + (m - m)(x - c),$$

which immediately yields

$$\begin{aligned} f(x) &> f(c) + 0(x - c) \\ f(x) &> f(c). \end{aligned}$$

[2] Any positive number less than or equal to m would work, but the simplest choice is m itself.

Since both cases lead to $f(x) > f(c)$, a contradiction, we conclude that our initial supposition was false. We had assumed that $f'(c) > 0$, so now we conclude that $f'(c) \not> 0$.

Proof that the derivative is nonnegative. The proof that $f'(c) \not< 0$ is similar. In the blank space below, or on a separate sheet of paper, please write out your own complete proof of the fact that $f'(c) \not< 0$.

Conclusion of the case where the extremum is a maximum. Having shown that $f'(c) \not> 0$ and that $f'(c) \not< 0$, we conclude that $f'(c) = 0$. We have now proven Theorem 1 for local maxima.

Case where the extremum is a minimum

The proof where $f(c)$ is a local minimum is the exact same, but with $-f$ in place of f, since a local maximum of f will be a local minimum of $-f$.

Conclusion

Having proven for both cases—$f(c)$ a local maximum and $f(c)$ a local minimum—that $f'(c) = 0$, we conclude that $f'(c) = 0$ for any local extremum of a function f at an interior point c of f's domain. □

The foregoing proof is very wordy. That's because it is a narrative proof. Its ideas are presented more or less in the order in which the proofwriter would have encountered them. For example, we selected an arbitrary positive ε, only later to realize that we needed to choose $\varepsilon = m$. In a less narrative, more polished proof, we would simply say from the outset that $\varepsilon = m$. Here is a more concise edition of this proof. Please read through it carefully.

Alternative edition of the above proof of Theorem 1. Let f be a function with a local extremum at an interior point c of its domain, and suppose $f'(c)$ exists. Since a local maximum of f will be a local minimum of $-f$, we can suppose without loss of generality that $f(c)$ is a local maximum.[3]

Proof that the derivative is nonpositive. Suppose for contradiction that $f'(c) > 0$. Let $\varepsilon = f'(c)$, so that by the definition of a derivative there exists $\delta > 0$ such that for all x,

$$0 < |x - c| < \delta \quad \Longrightarrow \quad \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < f'(c). \tag{3}$$

By the definition of a local maximum, there exists an open interval (a, b) containing c such that $f(x) < f(c)$ for all $x \in (a, b)$.

Pick x such that $c < x < \min\{b, c + \delta\}$.

Since $x \in (a, b)$, by the definition of a local maximum, therefore $f(x) \leq f(c)$.

Since $0 < |x - c| < \delta$, by (3), we know

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < f'(c). \tag{4}$$

[3] Including the phrase "without loss of generality" allows us to prove the two cases—where $f(c)$ is a maximum versus where it is a minimum—at the same time, rather than separately.

Case 1. Suppose the expression in the absolute value bars in (4) is nonnegative. This means that

$$\begin{aligned}\frac{f(x)-f(c)}{x-c}-f'(c) &\geq 0\\ \frac{f(x)-f(c)}{x-c} &\geq f'(c)\\ f(x)-f(c) &\geq f'(c)(x-c)\\ f(x) &\geq f(c)+f'(c)(x-c)\\ f(x) &> f(c),\end{aligned}$$

since $f'(c)$ and $(x-c)$ are both strictly positive.

Case 2. Suppose the expression in the absolute value bars in (4) is negative. Then (4) becomes

$$\begin{aligned}-\left(\frac{f(x)-f(c)}{x-c}-f'(c)\right) &< f'(c)\\ \frac{f(x)-f(c)}{x-c}-f'(c) &> -f'(c)\\ \frac{f(x)-f(c)}{x-c} &> 0\\ f(x)-f(c) &> 0\\ f(x) &> f(c).\end{aligned}$$

Both cases yield $f(x) > f(c)$, a contradiction, since $f(x) \leq f(c)$. So $f'(c) \not> 0$.

Proof that the derivative is nonnegative. Suppose for contradiction that $f'(c) < 0$. Let $\varepsilon = -f'(c)$, so that by the definition of a derivative there exists $\delta > 0$ such that for all x,

$$0 < |x-c| < \delta \quad \Longrightarrow \quad \frac{f(x)-f(c)}{x-c}-f'(c) \ < -f'(c). \tag{5}$$

By the definition of a local maximum, there exists an open interval (a,b) containing c such that $f(x) < f(c)$ for all $x \in (a,b)$.

Pick x such that $\max\{a, c-\delta\} < x < c$.

Since $x \in (a,b)$, by the definition of a local maximum, therefore $f(x) \leq f(c)$.

Since $0 < |x-c| < \delta$, by (5) we know

$$\frac{f(x)-f(c)}{x-c}-f'(c) \ < -f'(c). \tag{6}$$

Case 1. Suppose the expression in the absolute value bars in (6) is nonnegative. Then (6) becomes

$$\begin{aligned}\frac{f(x)-f(c)}{x-c}-f'(c) &< -f'(c)\\ \frac{f(x)-f(c)}{x-c} &< 0\\ f(x)-f(c) &> 0\\ f(x) &> f(c),\end{aligned}$$

since $x - c$ is negative.

Case 2. Suppose the expression in the absolute value bars in (6) is negative. This means that

$$\begin{aligned} \frac{f(x) - f(c)}{x - c} - f'(c) &< 0 \\ \frac{f(x) - f(c)}{x - c} &< f'(c) \\ f(x) - f(c) &> f'(c)(x - c) \\ f(x) &> f(c) + f'(c)(x - c) \\ f(x) &> f(c), \end{aligned}$$

since $f'(c)(x - c)$, being the product of two strictly negative values, is itself strictly positive.

In both cases, we have $f(x) > f(c)$, a contradiction, since $f(x) \le f(c)$. So $f'(c) \not< 0$.

Conclusion. We have shown that $f'(c) \not> 0$ and that $f'(c) \not< 0$. Therefore, by the trichotomy property of the real numbers, we conclude that $f'(c) = 0$. □

Once you have read and understood both versions of the proof above, continue trying to prove the rest of the theorems.

Theorem 2. *If $f'(x) \ge 0$ for every $x \in (a, b)$, then $f(x)$ is increasing on the interval $[a, b]$.*

Theorem 3. *If $f'(x) \le 0$ for every $x \in (a, b)$, then $f(x)$ is decreasing on the interval $[a, b]$.*

Theorem 4. *If $f'(x)$ changes sign on either side of the point $x = c$, then a local extremum occurs at $x = c$.*

Theorem 5. *If $f''(c) > 0$, then $f(x)$ is concave up at $x = c$.*

Theorem 6. *If $f''(c) < 0$, then $f(x)$ is concave down at $x = c$.*

Theorem 7. *If $f'(c) = 0$ and $f''(c) > 0$, then $f(x)$ has a local minimum at $x = c$.*

Theorem 8. *If $f'(c) = 0$ and $f''(c) < 0$, then $f(x)$ has a local maximum at $x = c$.*