GreatHearts Irving

Remote Learning Packet

NB: Please keep all work produced this week. Details regarding how to turn in this work will be forthcoming.

April 20 - 24, 2020
Course: 11 Calculus I
Teacher(s): Mr. Simmons

Weekly Plan:

Monday, April 20
☐ Revise your proof of Fermat's Theorem.

Tuesday, April 21
☐ Extreme Value Theorem proof and diagram.

Wednesday, April 22
☐ Diagrams for Fermat's Theorem, Rolle's Theorem, and the MVT

Thursday, April 23
☐ Prove Rolle's Theorem.

Friday, April 24
☐ Prove the MVT.

Statement of Academic Honesty

I affirm that the work completed from the packet is mine and that I completed it independently.

____
Student Signature

I affirm that, to the best of my knowledge, my child completed this work independently

____
Parent Signature

Monday, April 20

I would like to apologize, because in the list of theorems that I sent you, there was a typo. In the hypotheses for two of the theorems, there were written nonstrict inequalities, but they should have been strict inequalities. I have corrected this in the new version.

If you have felt particularly challenged by these proofs, that's okay! I hope that this is an opportunity for you not to memorize a method and execute it perfectly, but rather to be challenged and to struggle with real mathematical problems. I highly encourage you to come to office hours (virtually) to ask questions about these problems. And feel free to email me as well!

This week's handout is a rewriting of those same theorems, along with one more, the Extreme Value Theorem. I apologize for all the changes, but I - along with the rest of you - am still adjusting to this new setup. My plan had been for us to study the relationship between graphs and their derivatives, for these theorems to arise naturally from our study, and then for us to prove these theorems rigorously - needless to say, we haven't had the liberty to pursue such mathematical play together, so our interactions with these theorems, for most of us, have probably felt more like getting hit by a bus.

That's okay.

In this week's handout, I have proven the Extreme Value Theorem and Fermat's Theorem (which you looked at last week), and your assignment for this week is to **check your work against mine, work to understand the proofs as they are written, and then prove for yourself both Rolle's Theorem and the Mean Value Theorem.** Leave the corollaries for later.

1. **Today,** you should revise your proof of Fermat's Theorem, using the proof I have provided as a guide. Feel free to write in your own language. I don't intend for you to copy my proof word for word. In fact, mine is probably clunky at times and could be rewritten to be more elegant.

Tuesday, April 21

1. Read carefully through the statement and proof of the Extreme Value Theorem.
2. Complete Exercise 1.

Wednesday, April 22

1. Read carefully through the statement and proof of Fermat's Theorem.
2. Complete Exercise 2.
3. Read carefully through the theorem statements for Rolle's Theorem and the Mean Value Theorem.
4. Complete Exercises 3 and 5.

Thursday, April 23

1. Complete Problem 4 of the handout.

Friday, April 24

1. Complete Problem 5 of the handout.

Important Derivative Theorems

Calculus I
Mr. Simmons

Theorem (Extreme Value Theorem)**.** *If a real-valued function f is continuous on the closed interval $[a, b]$, then f must attain a maximum and a minimum somewhere in $[a, b]$, each at least once.*

Exercise 1. Draw a clear diagram that represents the Extreme Value Theorem. You may even draw more than one diagram to illustrate the universality of the theorem.

Proof. Let f be a real-valued function that is continuous on the closed interval $[a, b]$. The theorem says that f must attain a maximum and a minimum. We will prove that it must attain a maximum, and the proof that it must attain a minimum is trivially similar.

For $f(x)$ to have a maximum value means that the function value gets that high and doesn't get any higher. There will be two parts to this proof. First we will show that there is a value that $f(x)$ doesn't get higher than. Then we will prove that it does get that high.

Part I. First we show that there must be a finite upper bound (a value that $f(x)$ doesn't get higher than). We will accomplish this by assuming that there is no such value, and then deriving a contradiction.

Suppose for contradiction that there is no finite upper bound on $f(x)$. This means that the function attains larger and larger values. There must be some point x_1 with $f(x_1) > 1$, and a point

x_2 with $f(x_2) > 2$, and a point x_3 with $f(x_3) > 3$, and in general for any positive integer n, a point x_n in $[a,b]$ such that $f(x_n) > n$.

Now look at this sequence (list) of points we just created:

$$x_1, x_2, x_3, \ldots, x_n, \ldots.$$

It's not clear that

$$\lim_{n\to\infty} x_n$$

exists, but we shall pick out certain x_n's from the list to form a subsequence (sub-list)

$$x_{n_1}, x_{n_2}, x_{n_3}, \ldots, x_{n_k}, \ldots,$$

such that

$$\lim_{k\to\infty} x_{n_k}$$

exists.[1] This will be useful at the end of the proof.

We find the subsequence by first splitting the interval $[a,b]$ in half down the middle and considering its two halves. There are infinitely many x_n's in $[a,b]$, so there must be infinitely many x_n's in either the left half or the right half (maybe both). Pick one for which that's the case, and call it $[a_1,b_1]$. Since there are infinitely many of the x_n's in $[a_1,b_1]$, we can pick one with $n \geq 1$ and call it x_{n_1}.

There are infinitely many x_n's in $[a_1,b_1]$, so there must be infinitely many x_n's in either its left or right half. Pick a half for which that's the case, and call it $[a_2,b_2]$. Select one of the infinitely many x_n's in $[a_2,b_2]$ that has $n \geq 2$ and call it x_{n_2}.

We continue in this way and end up with the sequence

$$x_{n_1}, x_{n_2}, x_{n_3}, \ldots, x_{n_k} \quad (n_k \geq k).$$

The intervals from which we chose the x_{n_k}'s were all in $[a,b]$, and their width approaches zero as k tends toward infinity,[2] so we conclude that

$$\lim_{k\to\infty} x_{n_k}$$

exists and is in $[a,b]$. Let's name this limit x. (I.e., let $x = \lim\limits_{k\to\infty} x_{n_k}$.)

We now observe that

$$f(x) = f\left(\lim_{k\to\infty} x_{n_k}\right) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{n\to\infty} f(x_n).$$

We can pass the limit outside of the function since we are given that the function is continuous. The last equality we can conclude since every x_{n_k} was chosen as one of the x_n's in the first place, and $n_k \geq k$.[3]

[1] To illustrate and perhaps clarify, if our sequence of x_n's was $\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{7}{8}, \frac{1}{16}, \frac{15}{16}, \frac{1}{32}, \frac{31}{32}, \ldots$, for example, our subsequence of x_{n_k}'s might be every other element from that list—namely the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$—because while our x_n's here wouldn't have a limit, our x_{n_k}'s would.

[2] If the width of $[a,b]$ is δ, then the width of $[a_1,b_1]$ is $\frac{\delta}{2}$, the width of $[a_2,b_2]$ is $\frac{\delta}{4}$, the width of $[a_3,b_3]$ is $\frac{\delta}{8}$, and in general the width of $[a_k,b_k]$ is $\frac{\delta}{2^k}$, which approaches zero as k tends toward infinity.

[3] If you are not satified with this justification, see if you can use the definition of a limit at infinity to work out a more rigorous justification.

Since we chose the x_n's originally such that the values $f(x_n)$ increased to infinity an n increased, we see that this limit cannot be finite. But then the above equality would then suggest that $f(x)$ does not exist. This contradicts the assumption that f is defined on the entire closed interval.

Having derived a contradiction, we see that our original assumption must have been false, so in fact there must be a finite upper bound.

Part II. We have now shown that there must be a finite upper bound. Let M be the smallest such upper bound.[4] It remains to be shown that there exists some point such that the value of the function there is M.

Since M is the least upper bound, this means that the function attains values closer and closer to M. That is, for some positive ε, there must be some point x_1 such that $f(x_1)$ is within ε of M, and a point x_2 such that $f(x_2)$ is within $\frac{\varepsilon}{2}$ of M, and a point x_3 such that $f(x_3)$ is within $\frac{\varepsilon}{3}$ of M, and in general for any positive integer n, a point x_n such that $f(x_n)$ is within $\frac{\varepsilon}{n}$ of M.[5]

Now look at this sequence of points

$$x_1, x_2, x_3, \ldots, x_n, \ldots.$$

It is not clear that

$$\lim_{n\to\infty} x_n$$

exists, but we shall find a subsequence

$$x_{n_1}, x_{n_2}, x_{n_3}, \ldots, x_{n_k}, \ldots$$

such that

$$\lim_{k\to\infty} x_{n_k}$$

exists.

To find this subsequence, split $[a,b]$ into two halves as before. Select a half that has infinitely many x_n's in it, choose an x_n from among them where $n \geq 1$, and call it x_{n_1}. As before, this continued process yields the subsequence

$$x_{n_1}, x_{n_2}, x_{n_3}, \ldots, x_{n_k}, \ldots \quad (n_k \geq k).$$

The intervals from which we chose the x_{n_k}'s were all in $[a,b]$, and they approach a width of zero as n tends toward infinity, so we conclude that

$$\lim_{k\to\infty} x_{n_k}$$

exists and is in $[a,b]$. Let's name this limit x.

[4] The reason we can do this depends on one of the defining properties of the real numbers. This property says there are no "holes" in the real numbers and is called the completeness property. To illustrate the need for this property, consider the function $f(x) = x^2 - 2$. Since $f(0) = -2$ and $f(2) = 2$, the Intermediate Value Theorem implies that there is some real number x between zero and 2 such that $f(x) = 0$. And there is, namely $\sqrt{2}$. But if we were working with rational instead of real numbers, this value would not exist because $\sqrt{2}$ is not a rational number, so the IVT would be false. Whether or not this makes sense, the point should be clear: the theorems of calculus rest on the theory of the real numbers. This theory is a rich and powerful theory, but it is more appropriately discussed in detail in an introductory real analysis class than a high school calculus class.

[5] Notice that these x_n's are not the same as those in Part I. I use the same variable to emphasize the similarity in the ideas we are using in the two parts.

We now observe, similarly to before, that

$$f(x) = f\left(\lim_{k\to\infty} x_{n_k}\right) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{n\to\infty} f(x_n),$$

with the same justifications as before.

Since we chose the x_n's originally such that the values $f(x_n)$ were approaching M, we know that

$$\lim_{n\to\infty} f(x_n) = M.$$

The above equality then suggests that

$$f(x) = M.$$

This shows what we were trying to prove, that the funtion f attains its maximum value at the point x in the interval $[a, b]$.

This concludes the proof. □

Theorem (Fermat's Theorem)**.** *If a local extremum occurs at an interior point, $x = c$, and $f'(c)$ exists, then*

$$f'(c) = 0.$$

Exercise 2. Draw a clear diagram that represents Fermat's Theorem. You may even draw more than one diagram to illustrate the universality of the theorem.

Proof. Let f be a function with a local extremum at an interior point c of its domain, and suppose $f'(c)$ exists. Since a local maximum of f will be a local minimum of $-f$, we can suppose without loss of generality that $f(c)$ is a local maximum.

Proof that the derivative is nonpositive. Suppose for contradiction that $f'(c) > 0$. Let $\varepsilon = f'(c)$, so that by the definition of a derivative there exists $\delta > 0$ such that for all x,

$$0 < |x - c| < \delta \quad \Longrightarrow \quad \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < f'(c). \tag{1}$$

By the definition of a local maximum, there exists an open interval (a, b) containing c such that $f(x) < f(c)$ for all $x \in (a, b)$.

Pick x such that $c < x < \min\{b, c + \delta\}$.

Since $x \in (a, b)$, by the definition of a local maximum, therefore $f(x) \le f(c)$.

Since $0 < |x - c| < \delta$, by (1), we know

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < f'(c). \tag{2}$$

Case 1. Suppose the expression in the absolute value bars in (2) is nonnegative. This means that

$$\begin{aligned}
\frac{f(x) - f(c)}{x - c} - f'(c) &\ge 0 \\
\frac{f(x) - f(c)}{x - c} &\ge f'(c) \\
f(x) - f(c) &\ge f'(c)(x - c) \\
f(x) &\ge f(c) + f'(c)(x - c) \\
f(x) &> f(c),
\end{aligned}$$

since $f'(c)$ and $(x - c)$ are both strictly positive.

Case 2. Suppose the expression in the absolute value bars in (2) is negative. Then (2) becomes

$$\begin{aligned}
-\left(\frac{f(x) - f(c)}{x - c} - f'(c) \right) &< f'(c) \\
\frac{f(x) - f(c)}{x - c} - f'(c) &> -f'(c) \\
\frac{f(x) - f(c)}{x - c} &> 0 \\
f(x) - f(c) &> 0 \\
f(x) &> f(c).
\end{aligned}$$

Both cases yield $f(x) > f(c)$, a contradiction, since $f(x) \le f(c)$. So $f'(c) \not> 0$.

Proof that the derivative is nonnegative. Suppose for contradiction that $f'(c) < 0$. Let $\varepsilon = -f'(c)$, so that by the definition of a derivative there exists $\delta > 0$ such that for all x,

$$0 < |x - c| < \delta \quad \Longrightarrow \quad \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < -f'(c). \tag{3}$$

By the definition of a local maximum, there exists an open interval (a, b) containing c such that $f(x) < f(c)$ for all $x \in (a, b)$.

Pick x such that $\max\{a, c-\delta\} < x < c$.
Since $x \in (a,b)$, by the definition of a local maximum, therefore $f(x) \le f(c)$.
Since $0 < |x-c| < \delta$, by (3) we know

$$\left| \frac{f(x) - f(c)}{x-c} - f'(c) \right| < -f'(c). \tag{4}$$

Case 1. Suppose the expression in the absolute value bars in (4) is nonnegative. Then (4) becomes

$$\begin{aligned} \frac{f(x)-f(c)}{x-c} - f'(c) &< -f'(c) \\ \frac{f(x)-f(c)}{x-c} &< 0 \\ f(x) - f(c) &> 0 \\ f(x) &> f(c), \end{aligned}$$

since $x - c$ is negative.

Case 2. Suppose the expression in the absolute value bars in (4) is negative. This means that

$$\begin{aligned} \frac{f(x)-f(c)}{x-c} - f'(c) &< 0 \\ \frac{f(x)-f(c)}{x-c} &< f'(c) \\ f(x) - f(c) &> f'(c)(x-c) \\ f(x) &> f(c) + f'(c)(x-c) \\ f(x) &> f(c), \end{aligned}$$

since $f'(c)(x-c)$, being the product of two strictly negative values, is itself strictly positive.

In both cases, we have $f(x) > f(c)$, a contradiction, since $f(x) \le f(c)$. So $f'(c) \not< 0$.

Conclusion. We have shown that $f'(c) \not> 0$ and that $f'(c) \not< 0$. Therefore, by the trichotomy property of the real numbers, we conclude that $f'(c) = 0$. □

Theorem (Rolle's Theorem). *If a real-valued function f is continuous on a closed interval $[a,b]$, differentiable on the open interval (a,b), and $f(a) = f(b)$, then there exists at least one c in the open interval (a,b) such that*

$$f'(c) = 0.$$

Exercise 3. Draw a clear diagram that represents Rolle's Theorem. You may even draw more than one diagram to illustrate the universality of the theorem.

Problem 4. Rolle's Theorem is a direct consequence of the Extreme Value Theorem and Fermat's Theorem. Prove Rolle's Theorem.

Theorem (Mean Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there exists some c in (a, b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Exercise 5. Draw a clear diagram that represents the Mean Value Theorem. You may even draw more than one diagram to illustrate the universality of the theorem.

Problem 6. The Mean Value Theorem is the generalized case of Rolle's Theorem. Using Rolle's Theorem, prove the Mean Value Theorem.

Corollary. *If $f'(x) = 0$ for all x in an open interval I, then*

$$f(x) = C$$

for all x in I, where C is a constant.

Corollary. *If $f'(x) = g'(x)$ at each point of an open interval I, then*

$$f(x) = g(x) + C$$

for all x in I, where C is a constant.

Corollary. *Suppose that f is continuous at each point of $[a, b]$ and differentiable at each point of (a, b).*

1. *If $f'(x) > 0$ at each point of (a, b), then $f(x)$ is increasing on $[a, b]$.*
2. *If $f'(x) < 0$ at each point of (a, b), then $f(x)$ is decreasing on $[a, b]$.*
3. *If $f''(x) > 0$ at each point of (a, b), then $f(x)$ is concave up on $[a, b]$.*
4. *If $f''(x) < 0$ at each point of (a, b), then $f(x)$ is concave down on $[a, b]$.*

Theorem. *If $f'(c) = 0$ and $f''(c) > 0$, then $f(x)$ has a local maximum at $x = c$.*

Theorem. *If $f'(c) = 0$ and $f''(c) < 0$, then $f(x)$ has a local minimum at $x = c$.*