GreatHearts Irving

Remote Learning Packet

NB: Please keep all work produced this week. Details regarding how to turn in this work will be forthcoming.

April 27 - May 1, 2020
Course: 11 Calculus I
Teacher(s): Mr. Simmons

Weekly Plan:

Monday, April 27
- [ ] Read my announcement on Google Classroom’s stream
- [ ] Story time!
- [ ] Read about suprema
- [ ] Complete Problem 1
- [ ] Read the definition of an intersection

Tuesday, April 28
- [ ] Read Ch. 11 of Spivak until p. 179

Wednesday, April 29
- [ ] Read Ch. 11, pp. 179-186

Thursday, April 30
- [ ] Read Ch. 11, pp. 186-189
- [ ] Optional proof (ungraded)

Friday, May 1
- [ ] Complete Problem 11.6 (“Problem 6”)

Monday, April 27

1. If you have not done so already, please read the long announcement that I posted on the stream (at 8am today).
2. Story time! If it's technologically feasible to do, please email me at least one sentence letting me know how you're doing, and tell me a fun story. I miss you (yes, you).

As a way of reviewing and solidifying some of the material from the past few weeks, we are going to be reading Spivak. Some of this is review, so if anyone is bored, I challenge you to prove each of Spivak's theorems yourself (before looking at Spivak's proof), and then completing as many as you can of the very fascinating problems he has at the end of each chapter. You'll enjoy them.

In reading Spivak, we are coming in at Ch. 11. He will employ two terms that we have not covered, but which will not take you longer than today's 40 minutes to become familiar with. In service of preparing us to read Chapter 11, spend today following these instructions:

3. Read about suprema (and infima) on pp. 116-18, starting at the beginning of the chapter and stopping at the statement of Theorem 7-1.
4. On a separate sheet of paper, complete Problem 8.1 (he calls it Problem 1, but to clarify which chapter it's in, I'm calling it Problem 8.1) at the end of Ch. 8. (You will use this same piece of paper the whole week, turning in just this one item next Sunday.)
5. Read the very brief definition of "intersection" in the middle of p. 43 (in the paragraph that starts, "If f and g are any two functions…").

Remember to read for understanding. That means possibly pausing on a single sentence and thinking about it for a while, maybe drawing a few diagrams to help you understand it. Don't rush yourself, or you'll simply struggle more later.

Tuesday, April 28

1. Read Ch. 11, from the beginning of the chapter all the way through the proof of Rolle's Theorem on p. 179. Read for understanding. (At times, Spivak will write something like, "I leave the remainder of this proof as an exercise for the reader." Don't feel obliged to complete those problems (but of course feel free to!).)

Though Spivak doesn't label it as such, his Theorem 11.2 (he calls it Theorem 2, but to clarify which chapter it's in, I'm calling it Theorem 11.2) is Fermat's Theorem.

I am aware that he has a slightly different definition of "critical point" from the one we learned. He only mentions points where f prime is zero, whereas we had previously included also points where f prime is undefined (so long as f itself was defined). We will be going with Spivak's definition, as we will with everything from now on.

Wednesday, April 29

1. Read Ch. 11, until Theorem 11.5 (“Theorem 5”) on p. 186. (At times, Spivak will write something like, “I leave the remainder of this proof as an exercise for the reader.” Don’t feel obliged to complete those problems (but of course feel free to!).)

Thursday, April 30

1. Read Ch. 11, through p. 189.
2. (Optional, ungraded) Page 189 ends with a statement of L’Hopital’s Rule (LOE-pee-Tall’s). If you would like to, try to prove it. I’m happy to read your proof and give you feedback. (If you’re going to prove it yourself, careful not to glance at spoilers on p. 190.)

Friday, May 1

1. Finish reading Ch. 11 (not including the appendix on convexity and concavity, unless you want to). (At times, Spivak will write something like, “I leave the remainder of this proof as an exercise for the reader.” Don’t feel obliged to complete those problems (but of course feel free to!).)
2. Answer Problem 11.6 (“Problem 6”), writing on Monday’s sheet of paper.
3. If you have extra time of the 40 minutes allotted for math today, spend it mastering the vocabulary from Ch. 11

CHAPTER 8 LEAST UPPER BOUNDS

This chapter reveals the most important property of the real numbers. Nevertheless, it is merely a sequel to Chapter 7; the path which must be followed has already been indicated, and further discussion would be useless delay.

DEFINITION

A set A of real numbers is **bounded above** if there is a number x such that

$$x \geq a \quad \text{for every } a \text{ in } A.$$

Such a number x is called an **upper bound** for A.

Obviously A is bounded above if and only if there is a number x which is an upper bound for A (and in this case there will be lots of upper bounds for A); we often say, as a concession to idiomatic English, that "A has an upper bound" when we mean that there is a number which is an upper bound for A.

Notice that the term "bounded above" has now been used in two ways—first, in Chapter 7, in reference to functions, and now in reference to sets. This dual usage should cause no confusion, since it will always be clear whether we are talking about a set of numbers or a function. Moreover, the two definitions are closely connected: if A is the set $\{f(x): a \leq x \leq b\}$, then the function f is bounded above on $[a, b]$ if and only if the set A is bounded above.

The entire collection $\mathbf{R}$ of real numbers, and the natural numbers $\mathbf{N}$, are both examples of sets which are *not* bounded above. An example of a set which *is* bounded above is

$$A = \{x: 0 \leq x < 1\}.$$

To show that A is bounded above we need only name some upper bound for A, which is easy enough; for example, 138 is an upper bound for A, and so are 2, $1\frac{1}{2}$, $1\frac{1}{4}$, and 1. Clearly, 1 is the least upper bound of A; although the phrase just introduced is self-explanatory, in order to avoid any possible confusion (in particular, to ensure that we all know what the superlative of "less" means), we define this explicitly.

DEFINITION

A number x is a **least upper bound** of A if

(1) x is an upper bound of A,
and (2) if y is an upper bound of A, then $x \leq y$.

The use of the indefinite article "a" in this definition was merely a concession to temporary ignorance. Now that we have made a precise definition, it is easily seen that if x and y are both least upper bounds of A, then $x = y$. Indeed, in this case

$x \leq y$, since y is an upper bound, and x is a least upper bound,
and $y \leq x$, since x is an upper bound, and y is a least upper bound;

it follows that $x = y$. For this reason we speak of *the* least upper bound of A. The term **supremum** of A is synonymous and has one advantage. It abbreviates quite nicely to

$$\sup A \qquad \text{(pronounced "soup A")}$$

and saves us from the abbreviation

$$\text{lub } A$$

(which is nevertheless used by some authors).

There is a series of important definitions, analogous to those just given, which can now be treated more briefly. A set A of real numbers is **bounded below** if there is a number x such that

$$x \leq a \quad \text{for every } a \text{ in } A.$$

Such a number x is called a **lower bound** for A. A number x is the **greatest lower bound** of A if

(1) x is a lower bound of A,
and (2) if y is a lower bound of A, then $x \geq y$.

The greatest lower bound of A is also called the **infimum** of A, abbreviated

$$\inf A;$$

some authors use the abbreviation

$$\text{glb } A.$$

One detail has been omitted from our discussion so far—the question of which sets have at least one, and hence exactly one, least upper bound or greatest lower bound. We will consider only least upper bounds, since the question for greatest lower bounds can then be answered easily (Problem 2).

If A is not bounded above, then A has no upper bound at all, so A certainly cannot be expected to have a least upper bound. It is tempting to say that A does have a least upper bound if it has *some* upper bound, but, like the principle of mathematical induction, this assertion can fail to be true in a rather special way. If $A = \emptyset$, then A is bounded above. Indeed, any number x is an upper bound for $\emptyset$:

$$x \geq y \quad \text{for every } y \text{ in } \emptyset,$$

simply because there is no y in $\emptyset$. Since *every* number is an upper bound for $\emptyset$, there is surely no least upper bound for $\emptyset$. With this trivial exception however, our assertion is true—and very important, definitely important enough to warrant consideration of details. We are finally ready to state the last property of the real numbers which we need.

(P13) (The least upper bound property) If A is a set of real numbers, $A \neq \emptyset$, and A is bounded above, then A has a least upper bound.

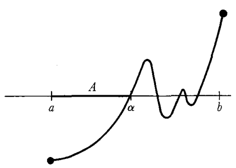


FIGURE 1

Property P13 may strike you as anticlimactic, but that is actually one of its virtues. To complete our list of basic properties for the real numbers we require no particularly abstruse proposition, but only a property so simple that we might feel foolish for having overlooked it. Of course, the least upper bound property is not really so innocent as all that; after all, it does *not* hold for the rational numbers $\mathbf{Q}$. For example, if A is the set of all rational numbers x satisfying $x^2 < 2$, then there is no *rational* number y which is an upper bound for A and which is less than or equal to every other *rational* number which is an upper bound for A. It will become clear only gradually how significant P13 is, but we are already in a position to demonstrate its power, by supplying the proofs which were omitted in Chapter 7.

THEOREM 7-1 If f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$, then there is some number x in $[a, b]$ such that $f(x) = 0$.

PROOF Our proof is merely a rigorous version of the outline developed at the end of Chapter 7—we will locate the smallest number x in $[a, b]$ with $f(x) = 0$.

Define the set A, shown in Figure 1, as follows:

$$A = \{x\colon a \leq x \leq b, \text{ and } f \text{ is negative on the interval } [a, x]\}.$$

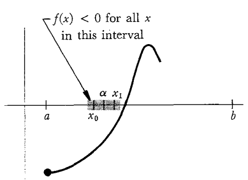


FIGURE 2

Clearly $A \neq \emptyset$, since a is in A; in fact, there is some $\delta > 0$ such that A contains all points x satisfying $a \leq x < a + \delta$; this follows from Problem 6-15, since f is continuous on $[a, b]$ and $f(a) < 0$. Similarly, b is an upper bound for A and, in fact, there is a $\delta > 0$ such that all points x satisfying $b - \delta < x \leq b$ are upper bounds for A; this also follows from Problem 6-15, since $f(b) > 0$.

From these remarks it follows that A has a least upper bound α and that $a < \alpha < b$. We now wish to show that $f(\alpha) = 0$, by eliminating the possibilities $f(\alpha) < 0$ and $f(\alpha) > 0$.

Suppose first that $f(\alpha) < 0$. By Theorem 6-3, there is a $\delta > 0$ such that $f(x) < 0$ for $\alpha - \delta < x < \alpha + \delta$ (Figure 2). Now there is some number x_0 in A which satisfies $\alpha - \delta < x_0 < \alpha$ (because otherwise α would not be the *least* upper bound of A). This means that f is negative on the whole interval $[a, x_0]$. But if x_1 is a number between α and $\alpha + \delta$, then f is also negative on the whole interval $[x_0, x_1]$. Therefore f is negative on the interval $[a, x_1]$, so x_1 is in A. But this contradicts the fact that a is an upper bound for A; our original assumption that $f(\alpha) < 0$ must be false.

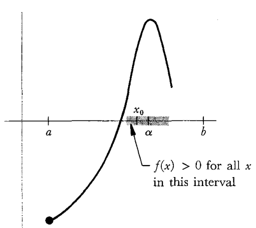


FIGURE 3

Suppose, on the other hand, that $f(\alpha) > 0$. Then there is a number $\delta > 0$ such that $f(x) > 0$ for $\alpha - \delta < x < \alpha + \delta$ (Figure 3). Once again we know that there is an x_0 in A satisfying $\alpha - \delta < x_0 < \alpha$; but this means that f is negative on $[a, x_0]$, which is impossible, since $f(x_0) > 0$. Thus the assumption

(12) $f(x) = \sin^2(\sin(\sin^2(x \sin^2 x^2))) \cdot \sin\left(\dfrac{x + \sin(x \sin x)}{x + \sin x}\right).$

By what criterion, you may feel impelled to ask, can such functions, especially a monstrosity like (12), be considered simple? The answer is that they can be built up from a few simple functions using a few simple means of combining functions. In order to construct the functions (9)–(12) we need to start with the "identity function" I, for which $I(x) = x$, and the "sine function" sin, whose value $\sin(x)$ at x is often written simply $\sin x$. The following are some of the important ways in which functions may be combined to produce new functions.

If f and g are any two functions, we can define a new function $f + g$, called the **sum** of f and g, by the equation

$$(f + g)(x) = f(x) + g(x).$$

Note that according to the conventions we have adopted, the domain of $f + g$ consists of all x for which "$f(x) + g(x)$" makes sense, i.e., the set of all x in both domain f and domain g. If A and B are any two sets, then $A \cap B$ (read "A intersect B" or "the intersection of A and B") denotes the set of x in both A and B; this notation allows us to write domain $(f + g)$ = domain $f \cap$ domain g.

In a similar vein, we define the **product** $f \cdot g$ and the **quotient** $\dfrac{f}{g}$ (or f/g) of f and g by

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

and

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$

Moreover, if g is a function and c is a number, we define a new function $c \cdot g$ by

$$(c \cdot g)(x) = c \cdot g(x).$$

This becomes a special case of the notation $f \cdot g$ if we agree that the symbol c should also represent the function f defined by $f(x) = c$; such a function, which has the same value for all numbers x, is called a **constant function.**

The domain of $f \cdot g$ is domain $f \cap$ domain g, and the domain of $c \cdot g$ is simply the domain of g. On the other hand, the domain of f/g is rather complicated—it may be written domain $f \cap$ domain $g \cap \{x\colon g(x) \neq 0\}$, the symbol $\{x\colon g(x) \neq 0\}$ denoting the set of numbers x such that $g(x) \neq 0$. In general, $\{x\colon \ldots\}$ denotes the set of all x such that "$\ldots$" is true. Thus $\{x\colon x^3 + 3 < 11\}$ denotes the set of all numbers x such that $x^3 < 8$, and consequently $\{x\colon x^3 + 3 < 11\} = \{x\colon x < 2\}$. Either of these symbols could just as well have been written using y everywhere instead of x. Variations of this notation are common, but hardly require any discussion. Any one can guess that $\{x > 0\colon x^3 < 8\}$ denotes the set of positive numbers whose cube is less than 8; it could be expressed more formally as $\{x\colon x > 0 \text{ and } x^3 < 8\}$.

CHAPTER 11 SIGNIFICANCE OF THE DERIVATIVE

One aim in this chapter is to justify the time we have spent learning to find the derivative of a function. As we shall see, knowing just a little about f' tells us a lot about f. Extracting information about f from information about f' requires some difficult work, however, and we shall begin with the one theorem which is really easy.

This theorem is concerned with the maximum value of a function on an interval. Although we have used this term informally in Chapter 7, it is worthwhile to be precise, and also more general.

DEFINITION Let f be a function and A a set of numbers contained in the domain of f. A point x in A is a **maximum point** for f on A, if

$$f(x) \geq f(y) \quad \text{for every } y \text{ in } A.$$

The number $f(x)$ itself is called the **maximum value** of f on A (and we also say that f "has its maximum value on A at x").

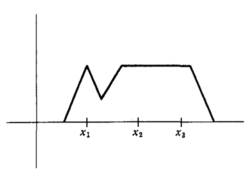


FIGURE 1

Notice that the maximum value of f on A could be $f(x)$ for several different x (Figure 1); in other words, a function f can have several different maximum points on A, although it can have at most one maximum value. Usually we shall be interested in the case where A is a closed interval $[a, b]$; if f is continuous, then Theorem 7-3 guarantees that f does indeed have a maximum value on $[a, b]$.

The definition of a minimum of f on A will be left to you. (One possible definition is the following: f has a minimum on A at x, if $-f$ has a maximum on A at x.)

We are now ready for a theorem which does not even depend upon the existence of least upper bounds.

THEOREM 1 Let f be any function defined on (a, b). If x is a maximum (or a minimum) point for f on (a, b), and f is differentiable at x, then $f'(x) = 0$. (Notice that we do not assume differentiability, or even continuity, of f at other points.)

PROOF Consider the case where f has a maximum at x. (Figure 2 illustrates the simple idea behind the whole argument—secants drawn through points to the left of $(x, f(x))$ have slopes ≥ 0, and secants drawn through points to the right of $(x, f(x))$ have slopes ≤ 0.) Analytically, this argument proceeds as follows.

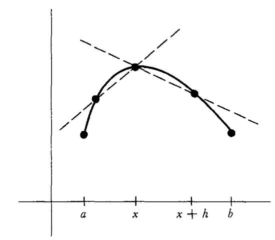


FIGURE 2

If h is any number such that $x + h$ is in (a, b), then

$$f(x) \geq f(x + h),$$

since f has a maximum on (a, b) at x. This means that

$$f(x + h) - f(x) \leq 0.$$

Thus, if $h > 0$ we have

$$\frac{f(x + h) - f(x)}{h} \leq 0,$$

and consequently

$$\lim_{h \to 0^+} \frac{f(x + h) - f(x)}{h} \leq 0.$$

On the other hand, if $h < 0$, we have

$$\frac{f(x + h) - f(x)}{h} \geq 0,$$

so

$$\lim_{h \to 0^-} \frac{f(x + h) - f(x)}{h} \geq 0.$$

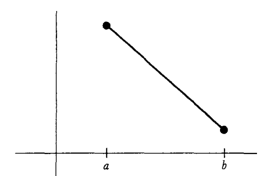


FIGURE 3

By hypothesis, f is differentiable at x, so these two limits must be equal, in fact equal to $f'(x)$. This means that

$$f'(x) \leq 0 \quad \text{and} \quad f'(x) \geq 0,$$

from which it follows that $f'(x) = 0$.

The case where f has a minimum at x is left to you (give a one-line proof). ▮

Notice (Figure 3) that we cannot replace (a, b) by $[a, b]$ in the statement of the theorem (unless we add to the hypothesis the condition that x is in (a, b)).

Since $f'(x)$ depends only on the values of f near x, it is almost obvious how to get a stronger version of Theorem 1. We begin with a definition which is illustrated in Figure 4.

DEFINITION Let f be a function, and A a set of numbers contained in the domain of f. A point x in A is a **local maximum** [**minimum**] **point** for f on A if there is some $\delta > 0$ such that x is a maximum [minimum] point for f on $A \cap (x - \delta, x + \delta)$.

THEOREM 2 If f is defined on (a, b) and has a local maximum (or minimum) at x, and f is differentiable at x, then $f'(x) = 0$.

PROOF You should see why this is an easy application of Theorem 1. ▮

The converse of Theorem 2 is definitely not true—it is possible for $f'(x)$ to be 0 even if x is not a local maximum or minimum point for f. The simplest example is provided by the function $f(x) = x^3$; in this case $f'(0) = 0$, but f has no local maximum or minimum anywhere.

Probably the most widespread misconceptions about calculus are concerned with the behavior of a function f near x when $f'(x) = 0$. The point made in the previous paragraph is so quickly forgotten by those who want the world to be simpler than it is, that we will repeat it: the converse of Theorem 2 is *not* true—the condition $f'(x) = 0$ does *not* imply that x is a local maximum or minimum point of f. Precisely for this reason, special terminology has been adopted to describe numbers x which satisfy the condition $f'(x) = 0$.

DEFINITION

A **critical point** of a function f is a number x such that

$$f'(x) = 0.$$

The number $f(x)$ itself is called a **critical value** of f.

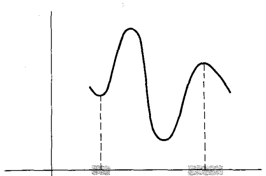


FIGURE 4

The critical values of f, together with a few other numbers, turn out to be the ones which must be considered in order to find the maximum and minimum of a given function f. To the uninitiated, finding the maximum and minimum value of a function represents one of the most intriguing aspects of calculus, and there is no denying that problems of this sort are fun (until you have done your first hundred or so).

Let us consider first the problem of finding the maximum or minimum of f on a closed interval $[a, b]$. (Then, if f is continuous, we can at least be sure that a maximum and minimum value exist.) In order to locate the maximum and minimum of f three kinds of points must be considered:

(1) The critical points of f in $[a, b]$.
(2) The end points a and b.
(3) Points x in $[a, b]$ such that f is not differentiable at x.

If x is a maximum point or a minimum point for f on $[a, b]$, then x must be in one of the three classes listed above: for if x is not in the second or third group, then x *is* in (a, b) and f *is* differentiable at x; consequently $f'(x) = 0$, by Theorem 1, and this means that x is in the first group.

If there are many points in these three categories, finding the maximum and minimum of f may still be a hopeless proposition, but when there are only a few critical points, and only a few points where f is not differentiable, the procedure is fairly straightforward: one simply finds $f(x)$ for each x satisfying $f'(x) = 0$, and $f(x)$ for each x such that f is not differentiable at x and, finally, $f(a)$ and $f(b)$. The biggest of these will be the maximum value of f, and the smallest will be the minimum. A simple example follows.

Suppose we wish to find the maximum and minimum value of the function

$$f(x) = x^3 - x$$

on the interval $[-1, 2]$. To begin with, we have

$$f'(x) = 3x^2 - 1,$$

so $f'(x) = 0$ when $3x^2 - 1 = 0$, that is, when

$$x = \sqrt{1/3} \quad \text{or} \quad -\sqrt{1/3}.$$

The numbers $\sqrt{1/3}$ and $-\sqrt{1/3}$ both lie in $[-1, 2]$, so the first group of candidates for the location of the maximum and the minimum is

$$(1) \quad \sqrt{1/3},\ -\sqrt{1/3}.$$

The second group contains the end points of the interval

$$(2) \quad -1,\ 2.$$

The third group is empty, since f is differentiable everywhere. The final step is to compute

$$\begin{aligned} f(\sqrt{1/3}) &= (\sqrt{1/3})^3 - \sqrt{1/3} = \tfrac{1}{3}\sqrt{1/3} - \sqrt{1/3} = -\tfrac{2}{3}\sqrt{1/3}, \\ f(-\sqrt{1/3}) &= (-\sqrt{1/3})^3 - (-\sqrt{1/3}) = -\tfrac{1}{3}\sqrt{1/3} + \sqrt{1/3} = \tfrac{2}{3}\sqrt{1/3}, \\ f(-1) &= 0, \\ f(2) &= 6. \end{aligned}$$

Clearly the minimum value is $-\frac{2}{3}\sqrt{1/3}$, occurring at $\sqrt{1/3}$, and the maximum value is 6, occurring at 2.

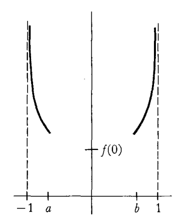


FIGURE 5

This sort of procedure, if feasible, will always locate the maximum and minimum value of a continuous function on a closed interval. If the function we are dealing with is not continuous, however, or if we are seeking the maximum or minimum on an open interval or the whole line, then we cannot even be sure beforehand that the maximum and minimum values exist, so all the information obtained by this procedure may say nothing. Nevertheless, a little ingenuity will often reveal the nature of things. In Chapter 7 we solved just such a problem when we showed that if n is even, then the function

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

has a minimum value on the whole line. This proves that the minimum value must occur at some number x satisfying

$$0 = f'(x) = nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_0.$$

If we can solve this equation, and compare the values of $f(x)$ for such x, we can actually find the minimum of f. One more example may be helpful. Suppose we wish to find the maximum and minimum, if they exist, of the function

$$f(x) = \frac{1}{1 - x^2}$$

on the open interval $(-1, 1)$. We have

$$f'(x) = \frac{2x}{(1-x^2)^2},$$

so $f'(x) = 0$ only for $x = 0$. We can see immediately that for x close to 1 or -1 the values of $f(x)$ become arbitrarily large, so f certainly does not have a maximum. This observation also makes it easy to show that f has a minimum at 0. We just note (Figure 5) that there will be numbers a and b, with

$$-1 < a < 0 \quad \text{and} \quad 0 < b < 1,$$

such that $f(x) > f(0)$ for

$$-1 < x \leq a \quad \text{and} \quad b \leq x < 1.$$

This means that the minimum of f on $[a, b]$ is the minimum of f on all of $(-1, 1)$. Now on $[a, b]$ the minimum occurs either at 0 (the only place where $f' = 0$), or at a or b, and a and b have already been ruled out, so the minimum value is $f(0) = 1$.

In solving these problems we purposely did not draw the graphs of $f(x) = x^3 - x$ and $f(x) = 1/(1 - x^2)$, but it is not cheating to draw the graph (Figure 6) as long as you do not rely solely on your picture to prove anything. As a matter of fact, we are now going to discuss a method of sketching the graph of a function that really gives enough information to be used in discussing maxima and minima—in fact we will be able to locate even *local* maxima and minima. This method involves consideration of the sign of $f'(x)$, and relies on some deep theorems.

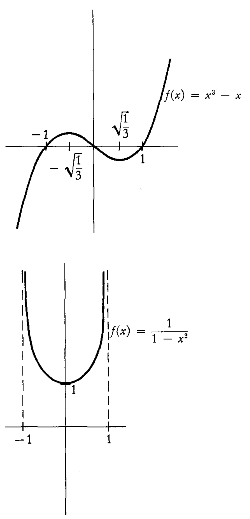


FIGURE 6

The theorems about derivatives which have been proved so far, always yield information about f' in terms of information about f. This is true even of Theorem 1, although this theorem can sometimes be used to determine certain information about f, namely, the location of maxima and minima. When the derivative was first introduced, we emphasized that $f'(x)$ is not $[f(x + h) - f(x)]/h$ for any particular h, but only a limit of these numbers as h approaches 0; this fact becomes painfully relevant when one tries to extract information about f from information about f'. The simplest and most frustrating illustration of the difficulties encountered is afforded by the following question: If $f'(x) = 0$ for all x, must f be a constant function? It is impossible to imagine how f could be anything else, and this conviction is strengthened by considering the physical interpretation—if the velocity of a particle is always 0, surely the particle must be standing still! Nevertheless it is difficult even to begin a proof that only the constant functions satisfy $f'(x) = 0$ for all x. The hypothesis $f'(x) = 0$ only means that

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = 0,$$

and it is not at all obvious how one can use the information about the limit to derive information about the function.

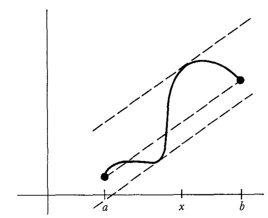


FIGURE 7

The fact that f is a constant function if $f'(x) = 0$ for all x, and many other facts of the same sort, can all be derived from a fundamental theorem, called the Mean Value Theorem, which states much stronger results. Figure 7 makes it plausible that if f is differentiable on $[a, b]$, then there is some x in (a, b) such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Geometrically this means that some tangent line is parallel to the line between $(a, f(a))$ and $(b, f(b))$. The Mean Value Theorem asserts that this is true—there is some x in (a, b) such that $f'(x)$, the instantaneous rate of change of f at x, is exactly equal to the average or "mean" change of f on $[a, b]$, this average change being $[f(b) - f(a)]/[b - a]$. (For example, if you travel 60 miles in one hour, then at some time you must have been traveling exactly 60 miles per hour.) This theorem is one of the most important theoretical tools of calculus—probably the deepest result about derivatives. From this statement you might conclude that the proof is difficult, but there you would be wrong—the hard theorems in this book have occurred long ago, in Chapter 7. It is true that if you try to prove the Mean Value Theorem yourself you will probably fail, but this is neither evidence that the theorem is hard, nor something to be ashamed of. The first proof of the theorem was an achievement, but today we can supply a proof which is quite simple. It helps to begin with a very special case.

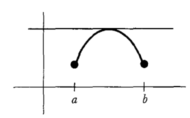


FIGURE 8

THEOREM 3 (ROLLE'S THEOREM) If f is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, then there is a number x in (a, b) such that $f'(x) = 0$.

PROOF It follows from the continuity of f on $[a, b]$ that f has a maximum and a minimum value on $[a, b]$.

Suppose first that the maximum value occurs at a point x in (a, b). Then $f'(x) = 0$ by Theorem 1, and we are done (Figure 8).

Suppose next that the minimum value of f occurs at some point x in (a, b). Then, again, $f'(x) = 0$ by Theorem 1 (Figure 9).

Finally, suppose the maximum and minimum values both occur at the end points. Since $f(a) = f(b)$, the maximum and minimum values of f are equal, so f is a constant function (Figure 10), and for a constant function we can choose any x in (a, b). ∎

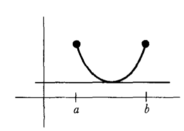


FIGURE 9

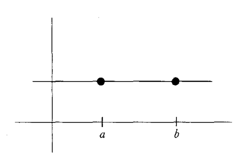


FIGURE 10

Notice that we really needed the hypothesis that f is differentiable everywhere on (a, b) in order to apply Theorem 1. Without this assumption the theorem is false (Figure 11).

You may wonder why a special name should be attached to a theorem as easily proved as Rolle's Theorem. The reason is, that although Rolle's Theorem is a special case of the Mean Value Theorem, it also yields a simple proof of the Mean Value Theorem. In order to prove the Mean Value

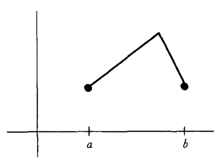


FIGURE 11

Theorem we will apply Rolle's Theorem to the function which gives the length of the vertical segment shown in Figure 12; this is the difference between $f(x)$, and the height at x of the line L between $(a, f(a))$ and $(b, f(b))$. Since L is the graph of

$$g(x) = \left[\frac{f(b) - f(a)}{b - a}\right](x - a) + f(a),$$

we want to look at

$$f(x) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a) - f(a).$$

As it turns out, the constant $f(a)$ is irrelevant.

THEOREM 4 (THE MEAN VALUE THEOREM) If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a number x in (a, b) such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

PROOF Let

$$h(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a).$$

Clearly, h is continuous on $[a, b]$ and differentiable on (a, b), and

$$\begin{aligned} h(a) &= f(a), \\ h(b) &= f(b) - \left[\frac{f(b) - f(a)}{b - a}\right](b - a) \\ &= f(a). \end{aligned}$$

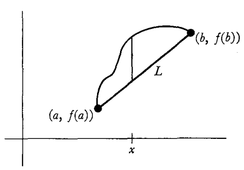


FIGURE 12

Consequently, we may apply Rolle's Theorem to h and conclude that there is some x in (a, b) such that

$$0 = h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

so that

$$f'(x) = \frac{f(b) - f(a)}{b - a}. \blacksquare$$

Notice that the Mean Value Theorem still fits into the pattern exhibited by previous theorems—information about f yields information about f'. This information is so strong, however, that we can now go in the other direction.

COROLLARY 1 If f is defined on an interval and $f'(x) = 0$ for all x in the interval, then f is constant on the interval.

PROOF Let a and b be any two points in the interval with $a \neq b$. Then there is some x

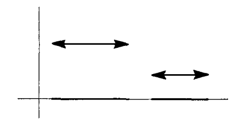


FIGURE 13

in (a, b) such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

But $f'(x) = 0$ for all x in the interval, so

$$0 = \frac{f(b) - f(a)}{b - a},$$

and consequently $f(a) = f(b)$. Thus the value of f at any two points in the interval is the same, i.e., f is constant on the interval. $\blacksquare$

Naturally, Corollary 1 does not hold for functions defined on two or more intervals (Figure 13).

COROLLARY 2 If f and g are defined on the same interval, and $f'(x) = g'(x)$ for all x in the interval, then there is some number c such that $f = g + c$.

PROOF For all x in the interval we have $(f - g)'(x) = f'(x) - g'(x) = 0$ so, by Corollary 1, there is a number c such that $f - g = c$. $\blacksquare$

The statement of the next corollary requires some terminology, which is illustrated in Figure 14.

DEFINITION A function f is **increasing** on an interval if $f(a) < f(b)$ whenever a and b are two numbers in the interval with $a < b$. The function f is **decreasing** on an interval if $f(a) > f(b)$ for all a and b in the interval with $a < b$. (We often say simply that f is increasing or decreasing, in which case the interval is understood to be the domain of f.)

COROLLARY 3 If $f'(x) > 0$ for all x in an interval, then f is increasing on the interval; if $f'(x) < 0$ for all x in the interval, then f is decreasing on the interval.

PROOF Consider the case where $f'(x) > 0$. Let a and b be two points in the interval with $a < b$. Then there is some x in (a, b) with

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

But $f'(x) > 0$ for all x in (a, b), so

$$\frac{f(b) - f(a)}{b - a} > 0.$$

Since $b - a > 0$ it follows that $f(b) > f(a)$.

The proof when $f'(x) < 0$ for all x is left to you. $\blacksquare$

Notice that although the converses of Corollary 1 and Corollary 2 are true (and obvious), the converse of Corollary 3 is not true. If f is increasing, it is easy to see that $f'(x) \geq 0$ for all x, but the equality sign might hold for some x (consider $f(x) = x^3$).

Corollary 3 provides enough information to get a good idea of the graph of a function with a minimal amount of point plotting. Consider, once more, the function $f(x) = x^3 - x$. We have

$$f'(x) = 3x^2 - 1.$$

We have already noted that $f'(x) = 0$ for $x = \sqrt{1/3}$ and $x = -\sqrt{1/3}$, and it is also possible to determine the sign of $f'(x)$ for all other x. Note that $3x^2 - 1 > 0$ precisely when

$$3x^2 > 1,$$
$$x^2 > \tfrac{1}{3},$$
$$x > \sqrt{1/3} \quad \text{or} \quad x < -\sqrt{1/3};$$

thus $3x^2 - 1 < 0$ precisely when

$$-\sqrt{1/3} < x < \sqrt{1/3}.$$

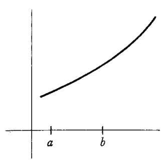
Thus f is increasing for $x < -\sqrt{1/3}$, decreasing between $-\sqrt{1/3}$ and $\sqrt{1/3}$, and once again increasing for $x > \sqrt{1/3}$. Combining this information with the following facts

(1) $f(-\sqrt{1/3}) = \frac{2}{3}\sqrt{1/3}$,
$\quad f(\sqrt{1/3}) = -\frac{2}{3}\sqrt{1/3}$,
(2) $f(x) = 0$ for $x = -1, 0, 1$,
(3) $f(x)$ gets large as x gets large, and large negative as x gets large negative,

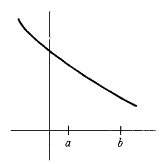
it is possible to sketch a pretty respectable approximation to the graph (Figure 15).

By the way, notice that the intervals on which f increases and decreases could have been found without even bothering to examine the sign of f'. For example, since f' is continuous, and vanishes only at $-\sqrt{1/3}$ and $\sqrt{1/3}$, we know that f' always has the same sign on the interval $(-\sqrt{1/3}, \sqrt{1/3})$. Since $f(-\sqrt{1/3}) > f(\sqrt{1/3})$, it follows that f decreases on this interval. Similarly, f' always has the same sign on $(\sqrt{1/3}, \infty)$ and $f(x)$ is large for large x, so f must be increasing on $(\sqrt{1/3}, \infty)$. Another point worth noting: If f' is continuous, then the sign of f' on the interval between two adjacent critical points can be determined simply by finding the sign of $f'(x)$ for any *one* x in this interval.

Our sketch of the graph of $f(x) = x^3 - x$ contains sufficient information to allow us to say with confidence that $-\sqrt{1/3}$ is a local maximum point, and $\sqrt{1/3}$ a local minimum point. In fact, we can give a general scheme for



(a) an increasing function



(b) a decreasing function

FIGURE 14

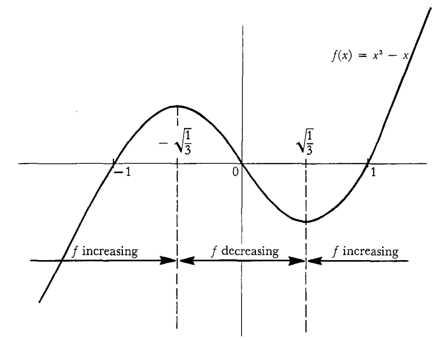


FIGURE 15

deciding whether a critical point is a local maximum point, a local minimum point, or neither (Figure 16):

(1) if $f' > 0$ in some interval to the left of x and $f' < 0$ in some interval to the right of x, then x is a local maximum point.
(2) if $f' < 0$ in some interval to the left of x and $f' > 0$ in some interval to the right of x, then x is a local minimum point.
(3) if f' has the same sign in some interval to the left of x as it has in some interval to the right, then x is neither a local maximum nor a local minimum point.

(There is no point in memorizing these rules—you can always draw the pictures yourself.)

The polynomial functions can all be analyzed in this way, and it is even possible to describe the general form of the graph of such functions. To begin,

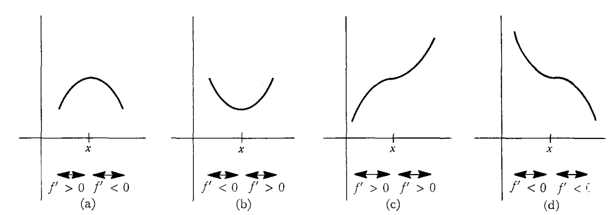


FIGURE 16

we need a result already mentioned in Problem 3-7: If

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0,$$

then f has at most n "roots," i.e., there are at most n numbers x such that $f(x) = 0$. Although this is really an algebraic theorem, calculus can be used to give an easy proof. Notice that if x_1 and x_2 are roots of f (Figure 17), so that $f(x_1) = f(x_2) = 0$, then by Rolle's Theorem there is a number x between x_1 and x_2 such that $f'(x) = 0$. This means that if f has k different roots $x_1 < x_2 < \cdots < x_k$, then f' has at least $k - 1$ different roots: one between x_1 and x_2, one between x_2 and x_3, etc. It is now easy to prove by induction that a polynomial function

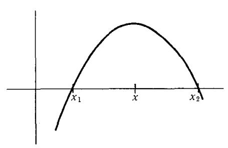


FIGURE 17

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$$

has at most n roots: The statement is surely true for $n = 1$, and if we assume that it is true for n, then the polynomial

$$g(x) = b_{n+1}x^{n+1} + b_nx^n + \cdots + b_0$$

could not have more than $n + 1$ roots, since if it did, g' would have more than n roots.

With this information it is not hard to describe the graph of

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0.$$

The derivative, being a polynomial function of degree $n - 1$, has at most $n - 1$ roots. Therefore f has at most $n - 1$ critical points. Of course, a critical point is not necessarily a local maximum or minimum point, but at any rate, if a and b are adjacent critical points of f, then f' will remain either positive or negative on (a, b), since f' is continuous; consequently, f will be either increasing or decreasing on (a, b). Thus f has at most n regions of decrease or increase.

As a specific example, consider the function

$$f(x) = x^4 - 2x^2.$$

Since

$$f'(x) = 4x^3 - 4x = 4x(x - 1)(x + 1),$$

the critical points of f are -1, 0, and 1, and

$$f(-1) = -1,$$
$$f(0) = 0,$$
$$f(1) = -1.$$

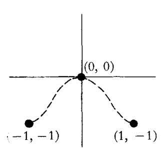


FIGURE 18

The behavior of f on the intervals between the critical points can be determined by one of the methods mentioned before. In particular, we could determine the sign of f' on these intervals simply by examining the formula for $f'(x)$. On the other hand, from the three critical values alone we can see (Figure 18) that f increases on $(-1, 0)$ and decreases on $(0, 1)$. To determine the sign of f' on $(-\infty, -1)$ and $(1, \infty)$ we can compute

$$f'(-2) = 4 \cdot (-2)^3 - 4 \cdot (-2) = -24,$$
$$f'(2) = 4 \cdot 2^3 - 4 \cdot 2 = 24,$$

and conclude that f is decreasing on $(-\infty, -1)$ and increasing on $(1, \infty)$. These conclusions also follow from the fact that $f(x)$ is large for large x and for large negative x.

We can already produce a good sketch of the graph; two other pieces of information provide the finishing touches (Figure 19). First, it is easy to determine that $f(x) = 0$ for $x = 0, \pm\sqrt{2}$; second, it is clear that f is even, $f(x) = f(-x)$, so the graph is symmetric with respect to the vertical axis. The function $f(x) = x^3 - x$, already sketched in Figure 15, is odd, $f(x) = -f(-x)$, and is consequently symmetric with respect to the origin. Half the work of graph sketching may be saved by noticing these things in the beginning.

Several problems in this and succeeding chapters ask you to sketch the graphs of functions. In each case you should determine

(1) the critical points of f,
(2) the value of f at the critical points,
(3) the sign of f' in the regions between critical points (if this is not already clear),
(4) the numbers x such that $f(x) = 0$ (if possible),
(5) the behavior of $f(x)$ as x becomes large or large negative (if possible).

Finally, bear in mind that a quick check, to see whether the function is odd or even, may save a lot of work.

This sort of analysis, if performed with care, will usually reveal the basic shape of the graph, but sometimes there are special features which require a little more thought. It is impossible to anticipate all of these, but one piece of information is often very important. If f is not defined at certain points (for example, if f is a rational function whose denominator vanishes at some points), then the behavior of f near these points should be determined.

For example, consider the function

$$f(x) = \frac{x^2 - 2x + 2}{x - 1},$$

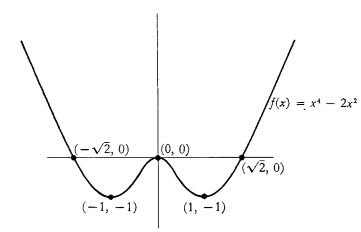


FIGURE 19

which is not defined at 1. We have

$$f'(x) = \frac{(x-1)(2x-2) - (x^2 - 2x + 2)}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}.$$

Thus

(1) the critical points of f are 0, 2.

Moreover,

(2) $f(0) = -2$,
$f(2) = 2$.

Because f is not defined on the whole interval $(0, 2)$, the sign of f' must be determined separately on the intervals $(0, 1)$ and $(1, 2)$, as well as on the intervals $(-\infty, 0)$ and $(2, \infty)$. We can do this by picking particular points in each of these intervals, or simply by staring hard at the formula for f'. Either way we find that

(3) $f'(x) > 0$ if $x < 0$,
$f'(x) < 0$ if $0 < x < 1$,
$f'(x) < 0$ if $1 < x < 2$,
$f'(x) > 0$ if $2 < x$.

Finally, we must determine the behavior of $f(x)$ as x becomes large or large negative, as well as when x approaches 1 (this information will also give us another way to determine the regions on which f increases and decreases). To examine the behavior as x becomes large we write

$$\frac{x^2 - 2x + 2}{x - 1} = x - 1 + \frac{1}{x-1};$$

clearly $f(x)$ is close to $x - 1$ (and slightly larger) when x is large, and $f(x)$ is close to $x - 1$ (but slightly smaller) when x is large negative. The behavior of f near 1 is also easy to determine; since

$$\lim_{x \to 1} (x^2 - 2x + 2) = 1 \neq 0,$$

the fraction

$$\frac{x^2 - 2x + 2}{x - 1}$$

becomes large as x approaches 1 from above and large negative as x approaches 1 from below.

All this information may seem a bit overwhelming, but there is only one way that it can be pieced together (Figure 20); be sure that you can account for each feature of the graph.

When this sketch has been completed, we might note that it looks like the

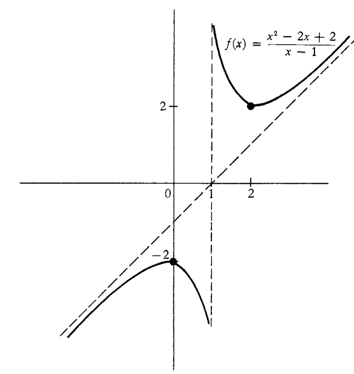


FIGURE 20

graph of an odd function shoved over 1 unit, and the expression

$$\frac{x^2 - 2x + 2}{x - 1} = \frac{(x-1)^2 + 1}{x - 1}$$

shows that this is indeed the case. However, this is one of those special features which should be investigated only after you have used the other information to get a good idea of the appearance of the graph.

Although the location of local maxima and minima of a function is always revealed by a detailed sketch of its graph, it is usually unnecessary to do so much work. There is a popular test for local maxima and minima which depends on the behavior of the function only at its critical points.

THEOREM 5 Suppose $f'(a) = 0$. If $f''(a) > 0$, then f has a local minimum at a; if $f''(a) < 0$, then f has a local maximum at a.

PROOF By definition,

$$f''(a) = \lim_{h \to 0} \frac{f'(a+h) - f'(a)}{h}.$$

Since $f'(a) = 0$, this can be written

$$f''(a) = \lim_{h \to 0} \frac{f'(a+h)}{h}.$$

Suppose now that $f''(a) > 0$. Then $f'(a + h)/h$ must be positive for sufficiently small h. Therefore:

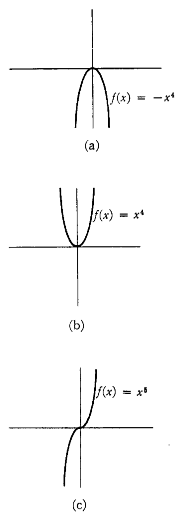


FIGURE 21

$$f'(a + h) \text{ must be positive for sufficiently small } h > 0$$
$$\text{and } f'(a + h) \text{ must be negative for sufficiently small } h < 0.$$

This means (Corollary 3) that f is increasing in some interval to the right of a and f is decreasing in some interval to the left of a. Consequently, f has a local minimum at a.

The proof for the case $f''(a) < 0$ is similar. ▮

Theorem 5 may be applied to the function $f(x) = x^3 - x$, which has already been considered. We have

$$f'(x) = 3x^2 - 1$$
$$f''(x) = 6x.$$

At the critical points, $-\sqrt{1/3}$ and $\sqrt{1/3}$, we have

$$f''(-\sqrt{1/3}) = -6\sqrt{1/3} < 0,$$
$$f''(\sqrt{1/3}) = 6\sqrt{1/3} > 0.$$

Consequently, $-\sqrt{1/3}$ is a local maximum point and $\sqrt{1/3}$ is a local minimum point.

Although Theorem 5 will be found quite useful for polynomial functions, the second derivative of many functions is so complicated that it is easier to consider the sign of the first derivative. Moreover, if a is a critical point of f it may happen that $f''(a) = 0$. In this case, Theorem 5 provides no information: it is possible that a is a local maximum point, a local minimum point, or neither, as shown (Figure 21) by the functions

$$f(x) = -x^4, \quad f(x) = x^4, \quad f(x) = x^5;$$

in each case $f'(0) = f''(0) = 0$, but 0 is a local maximum point for the first, a local minimum point for the second, and neither a local maximum nor minimum point for the third. This point will be pursued further in Part IV.

It is interesting to note that Theorem 5 automatically proves a partial converse of itself.

THEOREM 6 Suppose $f''(a)$ exists. If f has a local minimum at a, then $f''(a) \geq 0$; if f has a local maximum at a, then $f''(a) \leq 0$.

PROOF Suppose f has a local minimum at a. If $f''(a) < 0$, then f would also have a local maximum at a, by Theorem 5. Thus f would be constant in some interval containing a, so that $f''(a) = 0$, a contradiction. Thus we must have $f''(a) \geq 0$.

The case of a local maximum is handled similarly. ▮

(This partial converse to Theorem 5 is the best we can hope for: the $\geq$ and $\leq$ signs cannot be replaced by $>$ and $<$, as shown by the functions $f(x) = x^4$ and $f(x) = -x^4$.)

The remainder of this chapter deals, not with graph sketching, or maxima and minima, but with three consequences of the Mean Value Theorem. The

first is a simple, but very beautiful, theorem which plays an important role in Chapter 15, and which also sheds light on many examples which have occurred in previous chapters.

THEOREM 7 Suppose that f is continuous at a, and that $f'(x)$ exists for all x in some interval containing a, except perhaps for $x = a$. Suppose, moreover, that $\lim_{x \to a} f'(x)$ exists. Then $f'(a)$ also exists, and

$$f'(a) = \lim_{x \to a} f'(x).$$

PROOF By definition,

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

For sufficiently small $h > 0$ the function f will be continuous on $[a, a + h]$ and differentiable on $(a, a + h)$ (a similar assertion holds for sufficiently small $h < 0$). By the Mean Value Theorem there is a number α_h in $(a, a + h)$ such that

$$\frac{f(a + h) - f(a)}{h} = f'(\alpha_h).$$

Now α_h approaches a as h approaches 0, because α_h is in $(a, a + h)$; since $\lim_{x \to a} f'(x)$ exists, it follows that

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} f'(\alpha_h) = \lim_{x \to a} f'(x).$$

(It is a good idea to supply a rigorous ε-δ argument for this final step, which we have treated somewhat informally.) ▮

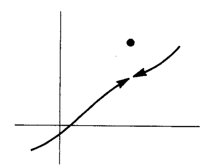


FIGURE 22

Even if f is an everywhere differentiable function, it is still possible for f' to be discontinuous. This happens, for example, if

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

According to Theorem 7, however, the graph of f' can never exhibit a discontinuity of the type shown in Figure 22. Problem 55 outlines the proof of another beautiful theorem which gives further information about the function f', and Problem 56 uses this result to strengthen Theorem 7.

The next theorem, a generalization of the Mean Value Theorem, is of interest mainly because of its applications.

THEOREM 8 (THE CAUCHY MEAN VALUE THEOREM) If f and g are continuous on $[a, b]$ and differentiable on (a, b), then there is a number x in (a, b) such that

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x).$$

(If $g(b) \neq g(a)$, and $g'(x) \neq 0$, this equation can be written

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x)}{g'(x)}.$$

Notice that if $g(x) = x$ for all x, then $g'(x) = 1$, and we obtain the Mean Value Theorem. On the other hand, applying the Mean Value Theorem to f and g separately, we find that there are x and y in (a, b) with

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x)}{g'(y)};$$

but there is no guarantee that the x and y found in this way will be equal. These remarks may suggest that the Cauchy Mean Value Theorem will be quite difficult to prove, but actually the simplest of tricks suffices.)

PROOF Let

$$h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)].$$

Then h is continuous on $[a, b]$, differentiable on (a, b), and

$$h(a) = f(a)g(b) - g(a)f(b) = h(b).$$

It follows from Rolle's Theorem that $h'(x) = 0$ for some x in (a, b), which means that

$$0 = f'(x)[g(b) - g(a)] - g'(x)[f(b) - (a)]. \blacksquare$$

The Cauchy Mean Value Theorem is the basic tool needed to prove a theorem which facilitates evaluation of limits of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)},$$

when

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0.$$

In this case, Theorem 5-3 is of no use. Every derivative is a limit of this form, and computing derivatives frequently requires a great deal of work. If some derivatives are known, however, many limits of this form can now be evaluated easily.

THEOREM 9 (L'HÔPITAL'S RULE) Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0,$$

and suppose also that $\lim_{x \to a} f'(x)/g'(x)$ exists. Then $\lim_{x \to a} f(x)/g(x)$ exists, and

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

(Notice that Theorem 7 is a special case.)

PROOF The hypothesis that $\lim_{x \to a} f'(x)/g'(x)$ exists contains two implicit assumptions:

(1) there is an interval $(a - \delta, a + \delta)$ such that $f'(x)$ and $g'(x)$ exist for all x in $(a - \delta, a + \delta)$ except, perhaps, for $x = a$,
(2) in this interval $g'(x) \neq 0$ with, once again, the possible exception of $x = a$.

On the other hand, f and g are not even assumed to be defined at a. If we define $f(a) = g(a) = 0$ (changing the previous values of $f(a)$ and $g(a)$, if necessary), then f and g are continuous at a. If $a < x < a + \delta$, then the Mean Value Theorem and the Cauchy Mean Value Theorem apply to f and g on the interval $[a, x]$ (and a similar statement holds for $a - \delta < x < a$). First applying the Mean Value Theorem to g, we see that $g(x) \neq 0$, for if $g(x) = 0$ there would be some x_1 in (a, x) with $g'(x_1) = 0$, contradicting (2). Now applying the Cauchy Mean Value Theorem to f and g, we see that there is a number α_x in (a, x) such that

$$[f(x) - 0]g'(\alpha_x) = [g(x) - 0]f'(\alpha_x)$$

or

$$\frac{f(x)}{g(x)} = \frac{f'(\alpha_x)}{g'(\alpha_x)}.$$

Now α_x approaches a as x approaches a, because α_x is in (a, x); since $\lim_{y \to a} f'(y)/g'(y)$ exists, it follows that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(\alpha_x)}{g'(\alpha_x)} = \lim_{y \to a} \frac{f'(y)}{g'(y)}.$$

(Once again, the reader is invited to supply the details of this part of the argument.) $\blacksquare$

PROBLEMS

1. For each of the following functions, find the maximum and minimum values on the indicated intervals, by finding the points in the interval where the derivative is 0, and comparing the values at these points with the values at the end points.

(i) $f(x) = x^3 - x^2 - 8x + 1$ on $[-2, 2]$.
(ii) $f(x) = x^5 + x + 1$ on $[-1, 1]$.
(iii) $f(x) = 3x^4 - 8x^3 + 6x^2$ on $[-\frac{1}{2}, \frac{1}{2}]$.
(iv) $f(x) = \dfrac{1}{x^5 + x + 1}$ on $[-\frac{1}{2}, 1]$.
(v) $f(x) = \dfrac{x + 1}{x^2 + 1}$ on $[-1, \frac{1}{2}]$.
(vi) $f(x) = \dfrac{x}{x^2 - 1}$ on $[0, 5]$.

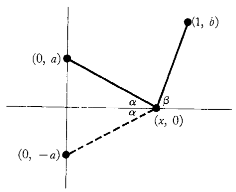


FIGURE 23

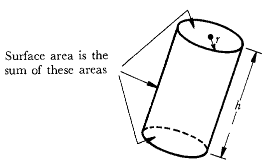


FIGURE 24

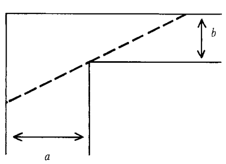


FIGURE 25

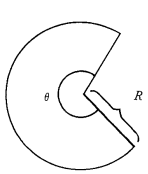


FIGURE 26

6. (a) Let (x_0, y_0) be a point of the plane, and let L be the graph of the function $f(x) = mx + b$. Find the point $\bar{x}$ such that the distance from (x_0, y_0) to $(\bar{x}, f(\bar{x}))$ is smallest. [Notice that minimizing this distance is the same as minimizing its square. This may simplify the computations somewhat.]

(b) Also find $\bar{x}$ by noting that the line from (x_0, y_0) to $(\bar{x}, f(\bar{x}))$ is perpendicular to L.

(c) Find the distance from (x_0, y_0) to L, i.e., the distance from (x_0, y_0) to $(\bar{x}, f(\bar{x}))$. [It will make the computations easier if you first assume that $b = 0$; then apply the result to the graph of $f(x) - mx$ and the point $(x_0, y_0 - b)$.] Compare with Problem 4-22.

(d) Consider a straight line described by the equation $Ax + By + C = 0$ (Problem 4-7). Show that the distance from (x_0, y_0) to this line is $(Ax_0 + By_0 + C)/\sqrt{A^2 + B^2}$.

7. The previous Problem suggests the following question: What is the relationship between the critical points of f and those of f^2?

8. A straight line is drawn from the point $(0, a)$ to the horizontal axis, and then back to $(1, b)$, as in Figure 23. Prove that the total length is shortest when the angles α and β are equal. (Naturally you must bring a function into the picture: express the length in terms of x, where $(x, 0)$ is the point on the horizontal axis. The dashed line in Figure 23 suggests an alternative geometric proof; in either case the problem can be solved without actually finding the point $(x, 0)$.)

9. Prove that of all rectangles with given perimeter, the square has the greatest area.

10. Find, among all right circular cylinders of fixed volume V, the one with smallest surface area (counting the areas of the faces at top and bottom, as in Figure 24).

11. A right triangle with hypotenuse of length a is rotated about one of its legs to generate a right circular cone. Find the greatest possible volume of such a cone.

12. Two hallways, of widths a and b, meet at right angles (Figure 25). What is the greatest possible length of a ladder which can be carried horizontally around the corner?

13. A garden is to be designed in the shape of a circular sector (Figure 26), with radius R and central angle θ. The garden is to have a fixed area A. For what value of R and θ (in radians) will the length of the fencing around the perimeter be minimized?

14. Show that the sum of a number and its reciprocal is at least 2.

15. Find the trapezoid of largest area that can be inscribed in a semicircle of radius a, with one base lying along the diameter.