GreatHearts
Irving

# Remote Learning Packet

*Please submit scans of written work in Google Classroom at the end of the week.*

**May 4-8, 2020**
**Course**: 11 Calculus I
**Teacher(s)**: Mr. Simmons

**Weekly Plan**:

Monday, May 4
- ☐ Read "The Big Picture"
- ☐ Read "Review Overview"
- ☐ Week 6: Practice Problems

Tuesday, May 5
- ☐ Week 6: Conceptual Questions 1 and 2

Wednesday, May 6
- ☐ Week 6: Conceptual Questions 3 and 4

Thursday, May 7
- ☐ Week 6: Conceptual Questions 5 and 6

Friday, May 8
- ☐ Attend office hours
- ☐ Catch up or review the week's work

## Monday, May 4

Dearest students of Calculus,

Thank you all for persevering during this unfortunate time. I'm sad that I can't be there with you to help us all understand better the mathematical truths we've been looking at.

This week begins our review of the whole year. We will be stepping back to get a big-picture look at things, and going back to solidify our knowledge of what we've learned. For today, please complete the following tasks:

1. Read "The Big Picture."
2. Read "Calculus I Review Overview."
3. Complete the practice problems for Week 6. (If you want to save completing these practice problems until after answering the conceptual problems, that's fine too.)

## Tuesday, May 5

1. Answer, in full, complete, grammatical sentences, the first and second conceptual questions for Week 6. I will not be surprised if this takes you the whole of today's 40 minutes. As it says in the instructions, you are writing as if you are teaching these concepts to someone who's never heard of them before. (If you wish, you may answer these two questions together, not in two separate answers.)

## Wednesday, May 6

1. Answer, in full, complete, grammatical sentences, the third and fourth conceptual questions for Week 6. I will not be surprised if this takes you the whole of today's 40 minutes. As it says in the instructions, you are writing as if you are teaching these concepts to someone who's never heard of them before. (If you wish, you may answer these two questions together, not in two separate answers.)

## Thursday, May 7

1. Answer, in full, complete, grammatical sentences, the fifth and sixth conceptual questions for Week 6. I will not be surprised if this takes you the whole of today's 40 minutes. As it says in the instructions, you are writing as if you are teaching these concepts to someone who's never heard of them before. (If you wish, you may answer these two questions together, not in two separate answers.)

# The Big Picture

*Mr. Simmons*
*11 Calculus I*

We have come to a point in our course where it is natural to turn our gaze backward and survey all that we've done. We're wrapping up our studies of derivatives, and next year you will begin with the study of integrals. Derivatives answer the first great question of Calculus: what is the slope of a tangent line? Integrals will answer the second: what is the area under a curve? The two are intimately related, but we will wait until next year to see how.

What we want to do now is threefold:

1. Solidify our understanding of the purpose and scope of Calculus.
2. Review the concepts of Calculus.
3. Practice the skills of Calculus.

This piece of writing is intended to achieve the first of these ends.

Let's step back and look at the big picture for a second. Calculus is the mathematical study of continuous change. What does that mean? Often, it means it's the mathematical study of physical change. The $x$-axis often represents time and the $y$-axis position. A curve, then, represents the motion of an object, and we can analyze that motion using derivatives. A derivative here represents instantaneous change. Under this interpretation, we can rephrase Fermat's theorem[1] to say that when you toss a ball up in the air, at its highest point, it is stationary for an instant, and knowing this is incredibly helpful in pinpointing exactly when it will reach that point. A bit of imagination let's us see how this theorem could be helpful to NASA for planning a rocket launch, for example. The theorems of Calculus are incredibly useful in physics, medicine, engineering, and many other fields.

But Calculus is the *mathematical* study of continuous change. We don't study it solely because it is useful in the sciences. You're not all going to be scientists. This is a math class, and we're learning to do math. That means we're learning to abstract ideas, form conjectures, and then prove those conjectures into theorems. What does all that mean?

To start with, I just used "abstract" as a verb. What does that mean? When you were young, you looked at a bunch of different apples, and you abstracted from your experience of all those different apples the general idea of *apple*. Then you started counting: one apple, two apples, three apples. Eventually you abstracted from all your counting the general ideas *one*, *two*, *three*, etc. Not "one apple," but just, *one*. Not "two apples," but just, *two*. We called these numbers. Then we started asking question about numbers, like "What's $5 + 2$?" or "What's $3928 \div 42$?" When we got bored answering simple questions like that, we started asking "What number would I multiply 36

[1] which, remember, says that if $f$ is defined on $(a, b)$ and has a local maximum (or minimum) at $x$, and $f$ is differentiable at $x$, then $f'(x) = 0$

by to get 2059?" We got tired of writing out long sentences like that, so we shortened it to "Say $36x = 2059$. What's $x$?" or even "Solve $36x = 2059$ for $x$."

(By the way, I didn't give you a definition of the verb "abstract." You're abstracting one from these sentences.)

Then you abstracted from the question, "What are $8 \times 2 + 10$, $8 \times 3 + 10$, and $8 \times 4 + 10$?," the general idea "$y = 8x + 10$." We called that a function. We noticed things were getting pretty darn abstract, so we came up with a whole new way to represent these ideas. Not just letters in place of numbers, but pictures in place of sentences. Graphs. It made it way easier to interpret functions.

Then you looked at a bunch of similar functions, like $y = 2x + 4$, $y = 5x - 3$, and $y = 39x + 9$, and you abstracted from them the general function equation

$$y = mx + b.$$

We called this a form. (Specifically, that one's called "slope–intercept form.") We abstracted some other forms too, like

$$y - y_1 = m\left(x - x_1\right),$$

$$y = ax^2 + bx + c,$$

$$y = \frac{a_p x^p + a_{p-1} x^{p-1} + \cdots + a_2 x^2 + a_1 x + a_0}{b_q x^q + b_{q-1} x^{q-1} + \cdots b_2 x^2 + b_1 x + b_0},$$

$$y = ab^x,$$

$$y = \sin\left(x\right),$$

and others. And we learned how to graph them all.

Now, in Calculus, we've come to a whole new level of abstraction. We've started saying interesting things about these functions. We started saying things like "A function whose graph is a smooth curve from here to here will have a highest and lowest point within that space." *Any* smooth function. We've gone from apples, to numbers, to variables, to functions, to classes of functions, to types of classes of functions, to statements about types of classes of functions. At first we thought the type we cared most about was the type called "continuous," because we thought those curves were always smooth, but then we discovered that some of them have ugly corners, like the absolute value function. So we got interested in a nicer type called "differentiable." Those functions were all very smooth. They reflected the real world, because things in the real world move around, speed up and slow down, in a smooth way. The statements we've been making about functions—at first functions in general, then continuous functions, then differentiable functions—are called theorems.

Theorems are at a very high level of abstraction. "If $f$ is defined on $(a, b)$ and has a local maximum (or minimum) at $x$, and $f$ is differentiable at $x$ then $f'(x) = 0$." What's $f$? Well, it's any differentiable function. So is it rational? Exponential? Trigonometric? If it's rational, is it linear? Quadratic? Cubic? Higher-order polynomial? Hyperbolic? If it's linear, is it $y = 2x$? $y = 31x + 5$? $y = -\frac{5}{6}x + \frac{2}{3}$? If it's, say, $y = 4x - 1$, then what number is $y$? Well, it depends on what number $x$ is. What number is $x$? It could be any number in the domain of the function. So, 4? 19485? $-0.009284$? $\pi$? What do $x$ and $y$ even represent? What are they the numbers of? Time versus position? Time versus money? Frequency versus amount? Are they just abstract variables, there simply to draw a beautiful graph?

There are so many layers of abstraction here. That's what makes Calculus so difficult. You might think theorems like this are too abstract to be useful. But they are *darn* useful. In fact, the more general the type of function that a theorem applies to (i.e., the more general the "if" part of the theorem), the more useful it is. And also the more specific the thing it says about that type of function (i. e., the more specific the "then" part of the theorem), the more useful it is. This theorem in particular helps us optimize a function, meaning find its maximum. That can be useful if you ever want to get the highest possible amount of something, which people often do (e. g., of money).[2]

So that's what a theorem is, a general, and therefore abstract, statement in math. What do I mean by saying that a math class teaches you to abstract ideas, form conjectures, and prove those conjectures into theorems? Well, forming conjectures is just the last step of abstraction as described above. A mathematician might notice a pattern: "All the differentiable functions I've ever seen have had $f'(x) = 0$ at their maxima. I wonder if that's true for *all* differentiable functions." That's a conjecture. That one in particular was proven by Pierre Fermat, causing it to be called a theorem (which just means a conjecture that's been proved). But there are still conjectures out there, yet to be proven. For example, the twin prime conjecture.[3] And there are an infinite number of conjectures yet to be made, maybe even an infinite number of interesting ones.

Once we've made a conjecture, a guess, we have to try to prove it. If it is proved, we call it a theorem. If it is disproved, we call it trash. (I'm kidding.) While we might use physical observations and intuitions to come up with conjectures, we can't depend on them to prove theorems. We prove theorems with deductive logic. Deductive logic doesn't rely on intuition, but on rigorous rules of inference.

For example, an intuitive understanding of a limit says that it's what the output approaches as the input approaches some value. If the input is interpreted as time and the output is interpreted as the position of an object, we would say that the limit is the place the object is going toward. In Calculus, it's essential to have this intuitive understanding, in order to be able to make guesses about which statements about limits might be true, and which ones might be false. But in math, if we're actually going to prove one of our guesses, we need a rigorous definition of a limit. "Going toward" isn't a rigorous mathematical concept. The formal, rigorous definition of a limit that we learned is that "The function $f$ approaches the limit $l$ near $a$ means: for every $\varepsilon > 0$ there is some $\delta > 0$ such that, for all $x$, if $0 < |x - a| < \delta$, then $|f(x) - l| < \varepsilon$." This is rigorous, because it relies not on any intuitions, but on terms that themselves have rigorous definitions, like "absolute value."

This is why math is so hard. It's so abstract. Just look at all those symbols. But that abstraction frees us from relying on physical intuitions, and it let's us prove things rigorously. That might seem silly, since we can just say, "But the informal definition *worked*." But rigor is what has allowed

[2] If you want to see a bunch of practical applications of calculus, look at page vi of your textbook. Finney loves practical applications.

[3] If you're interested, "twin primes" are any two primes that are only 2 apart. The first few twin prime pairs are 3 and 5, 5 and 7, 11 and 13, 17 and 19, 27 and 29, 29 and 31, 41 and 43.... You can already start to see they're getting rarer as we get higher. So do they eventually end? Is there a last twin prime pair? The twin prime conjecture states than there's no highest twin prime pair, that they just keep going, that no matter how high a number you pick, you'll be able to find a twin prime pair that's higher. "Beginning in 2007, two distributed computing projects, Twin Prime Search and PrimeGrid, have produced several record-largest twin primes. As of September 2018, the current largest twin prime pair known is $2996863034895 \cdot 2^{1290000} \pm 1$, with $388,342$ decimal digits. It was discovered in September 2016. There are $808,675,888,577,436$ twin prime pairs below $10^{18}$" (Wikipedia). But it's not enough just to show that they seem to keep going. Mathematicians want to prove they keep going. This conjecture has not been proven.

Calculus to advance far beyond what would have been possible otherwise. We need both intuition and rigor.

I want to make sure we're not missing the forest for the trees. Perhaps the theorems of Calculus seem to have been presented as abstract collections of symbols, with no real meaning. But of course, from what I've said above, that's not what they are. They do have meaning.

Rolle's theorem doesn't just say "If $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$, and $f(a) = f(b)$, then there is a number $x$ in $(a, b)$ such that $f'(x) = 0$"; it implies that if an object is in the same place now as it was a few minutes ago, then at some point between then and now it had to be stationary, even if only for an instant.

The Mean Value Theorem implies that if I'm here at one time and sixty miles away an hour later, then at some point, even if only for a moment, I had to be going 60 mph.

The theorems have meaning. The physical interpretations help us understand that meaning. They don't define it: the theorems are abstract mathematical truths. The variables $x$ and $y$ don't have to be time and position. That is often how the theorems are applied, and it can be useful to think of them that way, to see them as fitting into a larger picture. But always remember: in mathematics, we are studying eternal truths. The functions we're studying are abstract relations between quantities, independent from how those quantities are interpreted. The fact that they are true relations makes them incredibly useful in the sciences, but that's not primarily why we're studying them.

I hope this brief big-picture look at Calculus helps at least a little in conceptualizing and contextualizing the theorems of Calculus, as well as contextualizing Calculus among your other studies. As we come to the end of this academic year and look back on what we learned, we're not just going to review skills—how to apply the Chain Rule, for instance—but also make sure we have a firm understanding of each theorem and technique of Calculus as it relates to Calculus as a whole, the study of continuous change.

# Calculus I Review Overview

*Mr. Simmons*
*Calculus I*

## The Road Ahead

In these next three weeks (the three first full weeks of May), we're going to be reviewing Calculus I. (So that the numbers align with the week numbers for packets, we're going to start with "Week 6.") We will have an open-book assessment in Week 9 (the last week of the year).

This year, we've studied differential calculus (as opposed to integral calculus). Differential calculus answers the question of the slope of a tangent line; integral calculus answers the question of the area under a curve. The two will be connected in Calculus II, but for now let's take some time to make sure we understand the first of these.

The main idea of this year of Calculus has been to understand derivatives of functions. Derivatives are how we analyze continuous change. We learned years ago how to calculate the slope of a line. A line has a constant slope; a linear function has a constant rate of change. Whenever the independent variable changes by some amount, the dependent variable changes by a constant multiple of that amount.

But what about curves? Ever since ancient Greece, where $\pi$ was discovered, curves have been giving us trouble. That's where derivatives come in. If we look at a curve, we notice that we can draw a tangent line. Then we can measure the slope of that tangent line. Great! That tells us the rate of change of the curve right at the point where the tangent line touches it.

That's all that differential calculus is. It's just looking at curves and asking how sloped they are, how fast they change, and how fast that change is changing (and how fast *that* change is changing, and so on). But answering that question turned out not to be quite as easy as we might have guessed.

We ended up needing to base our definition of derivatives on a definition of limits. The derivative is the slope of the tangent line, but what's a tangent line? The best way we found to define the tangent line at a certain point (call it the tangent point) was for it to be the limit of secant lines (lines that go through the curve at two different points) as those two points get closer to the tangent point. So we defined the derivative at a point to be the limit of the secant line's slope as the secant line approaches the tangent line at that point. And there you have it: the slope of a curve, the instantaneous rate of change of a curvy function.

These are the main ideas behind differential calculus. In the next three weeks, we'll review the major concepts that help us understand these main ideas, and we'll practice again some of the skills we learned. Each week, I will give you two types of questions to answer: conceptual questions and practice questions. As an aid to answering these questions, I encourage you to look at the relevant chapter in your textbook.[1] As always, if you have questions, let me know. Below is a rough outline

[1] While Spivak's text is more rigorous and precise than Finney's, Finney can be more intuitive and easier to read for the purposes of a general overview, and you have easy access to it already. If you would like access to the relevant

of each week's main questions.

## Week 6: Functions, Limits, and Continuity

When we say we want to analyze continuous change, we're talking about continuous change of a function. So what's a function? We want to be able to define a tangent line, and I've hinted that we're going to use limits to do that. What's a limit? Finally, which kind of functions do we most want to look at? The answer is differentiable functions, which are a subclass of continuous functions… but what does "continuous" mean?

## Week 7: Derivatives

After understanding functions, limits, and continuity, we can finally give a little bit more form to our definition of derivatives. We know that the derivative is the slope of a tangent line, but how do we find that? Once we've found a derivative function, what can we do with it?

## Week 8: Derivatives and Graphs

Knowing the derivative of a function can be useful for graphing it. Let's understand how, and let's practice doing it.

excerpts from Spivak, please let me know by email.

# Week 6: Functions, Limits, and Continuity

*Mr. Simmons*
*Calculus I*

## Practice Problems

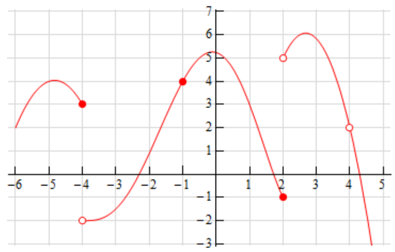
1. Given $f(x) = 3 - 5x - 2x^2$, evaluate
   (a) $f(4)$.
   (b) $f(0)$.
   (c) $f(-3)$.
   (d) $f(6 - t)$.
   (e) $f(7 - 4x)$.
   (f) $f(x + h)$.
2. Evaluate $\frac{f(x+h)-f(x)}{h}$ for
   (a) $f(x) = 4x - 9$.
   (b) $f(x) = \frac{2x}{3-x}$.
3. Determine the domain of each function:
   (a) $f(x) = 3x^2 - 2x + 1$
   (b) $f(x) = -x^2 - 4x + 7$
   (c) $f(x) = 2 + \sqrt{x^2 + 1}$
   (d) $f(x) = 5 - |x + 8|$
4. Find the following limits.
   (a) $\lim\limits_{x \to -5} \frac{x^2-25}{x^2+2x-15}$
   (b) $\lim\limits_{x \to 1} \frac{x^2-1}{x+1}$
   (c) $\lim\limits_{x \to 2} \frac{x^2-4}{x-2}$
   (d) $\lim\limits_{x \to 3} \frac{x^2-4}{x-2}$
   (e) $\lim\limits_{h \to 0} \frac{(a+h)^2-a^2}{h}$

5. Let $f(x) = \begin{cases} 7 - 4x & \text{if } x < 1 \\ x^2 + 2 & \text{if } x \geq 1 \end{cases}$. Find the following limits:

   (a) $\lim_{x \to -6} f(x)$

   (b) $\lim_{x \to 1} f(x)$

6. In the function graph below, determine where the function is discontinuous.




## Conceptual Questions

Answer the following questions in your own words. Try to avoid using symbols to the extent possible. Instead, write in full, complete, grammatical sentences. Answer these questions as if you're teaching these concepts to someone who's never heard of them before. That might mean giving examples, counterexamples, or analogies, for example. If you use any notation, it means explaining that notation. (This is the most important part of the review.)

1. In your own words, what is a function?
2. In your own words, what is a domain?
3. Exactly how are functions represented by equations?
4. Exactly how are functions represented by graphs?
5. In your own words, what is a limit?
6. In your own words, what is a continuity?