GreatHearts Irving

# Remote Learning Packet

*NB: Please keep all work produced this week. Details regarding how to turn in this work will be forthcoming.*

**April 13 - 17, 2020**
**Course**: 11 Precalculus
**Teacher(s)**: Mr. Simmons

**Weekly Plan**:

Monday, April 13
- [ ] Check the "Radians" handout answer key
- [ ] Begin the "More on Angles" handout

Tuesday, April 14
- [ ] Complete the "More on Angles" handout

Wednesday, April 15
- [ ] Check the "More on Angles" handout answer key
- [ ] Complete problem set 5.1: 7-39 odd

Thursday, April 16
- [ ] Check answers to problems set 5.1: 7-39 odd
- [ ] Complete problem set 5.1: 41-57 odd

Friday, April 17
- [ ] Check answers to problem set 5.1: 41-57 odd

## Statement of Academic Honesty

I affirm that the work completed from the packet is mine and that I completed it independently.

I affirm that, to the best of my knowledge, my child completed this work independently

______________________________
Student Signature

______________________________
Parent Signature

**Monday, April 13**

1. Check your answer in the “Radians” handout against my answer key.
2. Begin the “More on Angles” handout.

**Tuesday, April 14**

1. Complete the “More on Angles” handout.

**Wednesday, April 15**

1. Check your answers to the “More on Angles” handout against my answer key.
2. Complete problem set 5.1: 7-39 odd.

**Thursday, April 16**

1. Check answers to problems set 5.1: 7-39 odd in the back of the book.
2. Complete problem set 5.1: 41-57 odd.

**Friday, April 17**

1. Check answers to problem set 5.1: 41-57 odd in the back of the book.

# Radians – Answer Key

*Precalculus*
*Mr. Simmons*

*Read through this handout carefully and pause to think and respond when instructed.*

We got the unit called degrees (°) by dividing a full rotation into 360 equally sized angles and saying that each of those angles had measure 1°. Note that the number 360 was an arbitrary choice: we could have chosen 4, or 10, or any other counting number. But 360 is useful, because it is divisible by so many numbers (i.e., by 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, and 360 itself). Fun fact: 360 is therefore called a "highly composite" number. So are 12 and 60, which is why there are 12 inches in a foot and 60 minutes in an hour (and 60 seconds in a minute).[1]

But is there a less arbitrary way to measure an angle? When we look at an angle and wonder how big it is, we generally wonder, in an intuitive sense, how "far apart" the two rays are. A bigger angle will mean two rays that are "further apart." But, of course, the distance between the two rays is . . . zero. Always. Because they're touching (at the vertex).

So what do we do? Last handout, we pictured the standard position right angle as the swinging open of a door whose hinge is at the origin and whose knob swings from $(1, 0)$ to $(0, 1)$. This dynamic representation of an angle allows us to measure the angle not by asking how "far apart" the rays are (because that number will always be zero), but by asking how far the knob has swung. So let's trace the path of the knob.

In the space below (or, if you haven't been able to print out this document, then on a separate sheet of paper), sketch the aforementioned standard position right angle, the one represented by the door swinging open, with its vertex at the origin and rays that pass through the points $(1, 0)$ and $(0, 1)$:

Now put your pencil down at $(1, 0)$ and start drawing the unit circle counterclockwise, but stop once you get to $(0, 1)$. If we think of this angle as representing the opening of the door, then what you just traced is the path of motion of the doorknob.

[1] Just for fun, consider the pros and cons of the metric system of units versus the imperial system of units. Sure, the metric system simplifies everything to base 10, but there are good reasons to use highly composite numbers like 8, 12, and 36.

What you just drew is called an **arc**. An arc may be a portion of a full circle, a full circle, or even more than a full circle. The length of the arc around an entire circle is called the **circumference** of that circle. The arc you just drew is just a portion of the unit circle (one fourth, to be specific), and it has two **endpoints**: $(1, 0)$ and $(0, 1)$.

We say that the arc you just drew **subtends** our right angle, because the angle's rays go through the arc's endpoints, and the angle's vertex is the arc's circle's center (in this case, the origin). We said earlier that measuring the path of the knob, that is, measuring this arc, would help us measure the angle. Well, this arc has length one quarter the circumference of the unit circle. Take a moment to find out exactly what that is. Write your answer here (or on a separate sheet of paper):

There, we have a number! Can we say now that that's the measure of our angle? Why or why not? Write down your thoughts:

If we look at any given angle, intuitively we want to measure it, as we've said, by the length of the arc that subtends it. But the problem is, it is subtended by many arcs. An infinite number of arcs. Take, for instance, the right angle we were just considering. Depending on which circle you choose to draw over it (centered at the origin, of course), you could make it subtended by an arc of any length you like by simply making the the circle bigger or smaller. So how can we use arc lengths to measure angles?

One way to do it is simply, as we just did, to choose the unit circle every time. Given any angle, look at the arc on the unit circle that subtends that angle, and call that arc length the measure of the angle. A right angle is subtended by an arc of length—you calculated it earlier—$\frac{\pi}{2}$.[2] So we say that a right angle has measure $\frac{\pi}{2}$. Beautiful!

But picking the unit circle still feels a bit arbitrary. What's so special about the unit circle? Instead of picking a particular circle, shouldn't we pick an arbitrary circle? Angles aren't doors, they're connected rays. Unlike doors, rays are infinitely long, and there's nothing special about one point on the ray versus another (other than the vertex, but we already said we can't use that one). Is there any way that, given an angle we're trying to measure, we can come up with a measure, a number, based on arc length, without it mattering which arc we pick? Sounds crazy. Stop and think about it. Write your thoughts here (or, as before, if appropriate, on a separate sheet of paper):

[2] This is because the unit circle has circumference $2\pi r = 2\pi(1) = 2\pi$, one fourth of which is $\frac{\pi}{2}$.

Consider that, for any given angle you're trying to measure, as you choose bigger and bigger arcs, the circles that they are portions of will also be bigger and bigger. And what does it mean for a circle to be bigger? Before moving on, take a moment to recall the precise definition of a circle.

...

A circle is the set of all points equidistant from a center point, and we call that distance the circle's radius. So a circle is defined in terms of its radius. What it *means* for a circle to be "bigger" is that its radius is longer. So the bigger the arc you choose, the bigger the radius that goes along with it. If we choose a circle of radius 2, then instead of getting an arc of length $\frac{\pi}{2}$, we get an arc of length ... well, what's a quarter of this new circle's circumference? Write it down:

That's right, the new arc length is $\pi$. That's different from $\frac{\pi}{2}$. Sounds like a problem. But wait, if we got an arc of length $\frac{\pi}{2}$ when we had a circle of radius 1, and we got an arc of length $\pi$ when we had a circle of radius 2 .... Do we see a pattern? What's the pattern? Write down your thoughts:

Just as every integer, even if we don't write it as a ratio, is a ratio with a denominator of 1, so is every angle measure a ratio. When we pick the unit circle, we are choosing a denominator of 1; when we pick a circle of radius 2, we are choosing a denominator of 2; when we pick a circle of radius 3, we are choosing a denominator of 3; and so on. (I'm picking integers only because they're simple—you could pick literally any positive real number.) But

$$\frac{\frac{\pi}{2}\text{ arc length units}}{1\text{ radius unit}} = \frac{\pi\text{ arc length units}}{2\text{ radius units}} = \frac{\frac{3\pi}{2}\text{ arc length units}}{3\text{ radius units}} = \cdots = \frac{\pi}{2}.$$

So we can reasonably say, without any arbitrary choice, that the measure of a right angle is $\frac{\pi}{2}$. The ratio of arc length to radius doesn't change depending on which circle we pick. Given any angle, if you take an arc that subtends that angle and divide its length by its circle's radius, no matter which circle you choose, you always get the same answer. *The ratio of arc length to radius is constant for any given angle.* Sounds like we've found ourselves a consistent way to measure angles using arc lengths! This is particularly satisfying because it fits with our intuitive notion of angles as representing rotation. This way of measuring angles tells us quite straightforwardly how far the knob of our door has traveled, which is an intuitive way of picturing the size of the angle. Wonderful.

This measure of an angle—the one we get by dividing the arc length (of an arc that subtends the angle) by the radius of the circle (of which that arc is a portion)—is called the **radian measure** of an angle.

Teachnically speaking, the radian measure of an angle is stated in a unit called **radians**, where one radian is defined as the measure of the angle subtended by an arc on the unit circle of arc length 1—but very rarely does any mathematician write out the word "radians" or even the abbreviation "rad" next to the radian measure of an angle, and very rarely is that angle I just described, the one whose measure is 1 radian, ever used for anything. (It's also kind of ugly.[3]) Since the radian measure of an angle is gotten by dividing a length (an arc length) by another length (a radius), the length units cancel, leaving radians a dimensionless unit, or what mathematicians call a "pure number."

[3] Fun exercise: explain why an angle of radian measure 1 is ugly. Or, alternatively, argue that it is beautiful. Feel free to email me with responses.

That was a lot of work! As a way of solidifying the concepts covered in the preceding pages, go ahead and read through the rigorous statements of the definitions you just learned. While learning these definitions verbatim is not necessary, you should be able to give a complete, mathematically precise definition of each of these words from memory.

**Definition** (ARC)**.** An arc is a portion of a circle.

**Definition** (SUBTEND)**.** An arc subtends an angle if and only if the angle's two rays pass through the arc's two endpoints.

**Definition** (RADIAN MEASURE)**.** The radian measure of an angle is the ratio of the length of the arc that subtends the angle to the radius of the circle.

In other words, if $s$ is the length of an arc of a circle, and $r$ is the radius of the circle, then the central angle containing that arc measures $\frac{s}{r}$ radians. In a circle of radius 1, the radian measure corresponds to the length of the arc.

**Definition** (RADIAN)**.** One radian is the measure of the central angle of a circle such that the length of the arc between the initial side and the terminal side is equal to the radius of the circle.

*Complete the following exercises on a separate sheet of paper.*

**Exercise 1.** Find the radian measure of one third of a full rotation.

**Solution.** The radian measure of one third of a full rotation is

$$2\pi \cdot \frac{1}{3} = \frac{2\pi}{3}.$$

**Exercise 2.** Find the radian measure of three fourths of a full rotation.

**Solution.** The radian measure of three fourths of a full rotation is

$$2\pi \cdot \frac{3}{4} = \frac{3\pi}{2}.$$

**Exercise 3.** Remember that a conversion factor is a fraction, equal to one, that you multiply a measurement by to change its units. For example, to change 2 feet into inches, I multiply 2 feet by the conversion factor $\frac{12 \text{ in}}{1 \text{ ft}}$ to get

$$2 \text{ ft} \times \frac{12 \text{ in}}{1 \text{ ft}} = 24 \text{ in}.$$

The feet cancel, leaving only inches.

Come up with conversion factors to convert from degrees to radians and from radians to degrees.

**Solution.** There are multiple possible answers, but the standard conversion factors are as follows:

- for converting from degrees to radians:
$$\frac{\pi}{180^\circ}.$$
- for converting from radians to degrees:
$$\frac{180^\circ}{\pi}.$$

**Exercise 4.** Convert the radian measure $\frac{\pi}{6}$ into degrees.

**Solution.** In degrees, $\frac{\pi}{6}$ radians is

$$\frac{\pi}{6} \cdot \frac{180^\circ}{\pi} = 30^\circ.$$

**Exercise 5.** Convert the radian measure $3\pi$ into degrees.

**Solution.** In degrees, $3\pi$ radians is

$$3\pi \cdot \frac{180^\circ}{\pi} = 540^\circ.$$

**Exercise 6.** Convert $15^\circ$ into radians.

**Solution.** In radians, $15^\circ$ is

$$15^\circ \cdot \frac{\pi}{180^\circ} = \frac{\pi}{12}.$$

**Exercise 7.** Convert $126^\circ$ ito radians.

**Solution.** In radians, $126^\circ$ is

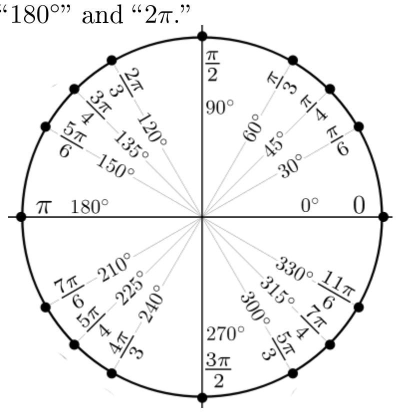
$$126^\circ \cdot \frac{\pi}{180^\circ} = \frac{7\pi}{10}.$$

**Exercise 8.** In a clear, neat diagram, draw the unit circle, and then sketch in standard position the following angles, given in portions of a full rotation. Then label, at the intersection of each angle's terminal side with the unit circle, the measure of that angle in both degrees and radians. (You may include the unit label "radians" or "rad" on the radian measure if you would like, but you needn't.)

$0$, $1$, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{6}$, $\frac{5}{6}$, $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, $\frac{7}{8}$, $\frac{1}{12}$, $\frac{5}{12}$, $\frac{7}{12}$, and $\frac{11}{12}$

For example, to draw and label an angle that is $\frac{1}{4}$ of a full rotation, you would draw the standard position right angle that we were dealing with all throughout this handout (the one represented by the swinging door) and label it, at the point $(0, 1)$, with the labels "90°" and "$\frac{\pi}{2}$."

**Solution.** Your diagram should look like the diagram below, but with the added labeling at $(1, 0)$ of "180°" and "$2\pi$."



# More on Angles

*Precalculus*
*Mr. Simmons*

*Read through this handout carefully and pause to think and respond when instructed.*

Any angle can be measured as being between $0°$ and $180°$. However, imagine a door, not the door we have been picturing for the last two handouts, but a revolving door. It starts out the same as the other door, with its hinge at the origin and its knob at $(1, 0)$, but instead of swinging open $90°$, it revolves in a complete circle three and a half times, ending its rotation with its knob at $(-1, 0)$. What angle does this rotation represent?

Three full turns of a circle plus a half would have an angle measure of $3.5 \cdot 360° = 1260°$. So we say that the angle represented by the abovedescribed rotation has an angle measure of $1260°$.

But now imagine that another person, who was not there when you were pushing the revolving door, walks up and sees the door with its knob at $(-1, 0)$. What angle measure do you think he would ascribe to the angle created by the rotation of the door?

He might assume that the door only swung half a revolution, rather than three and a half revolutions. Thinking this, he would surely say that the angle measure is $180°$. While he is inaccurately measuring the angle of $1260°$, he is accurately measuring what seems to be the same angle. We say that these two angles—of $1260°$ and of $180°$—are **coterminal**, and that the angle of $180°$ is the **reference angle** of the $1260°$-angle. Study these definitions:

**Definition** (COTERMINAL ANGLES)**.** Two angles are said to be coterminal if and only if they have the same terminal side when in standard position.

**Definition** (REFERENCE ANGLE)**.** An angle $t$'s reference angle is the smallest acute angle $t'$ formed by the terminal side of the angle $t$ and the horizontal axis.

Complete the following exercises:

**Exercise 1.** Find the least positive angle of measure $\theta$ that is coterminal with an angle measuring $800°$, where $0° \leq \theta < 360°$.

**Exercise 2.** Find an angle of measure $\alpha$ that is coterminal with an angle measuring $870°$, where $0° \leq \alpha < 360°$.

**Exercise 3.** Show the angle with measure $-45°$ on a circle and find a positive coterminal angle of measure $\alpha$ such that $0° \leq \alpha < 360°$.

**Exercise 4.** Find an angle of measure $\beta$ that is coterminal with an angle measuring $-300°$ such that $0° \leq \beta < 360°$.

**Exercise 5.** Find an angle of measure $\beta$ that is coterminal with an angle of measure $\frac{19\pi}{4}$, where $0 \leq \beta < 2\pi$.

**Exercise 6.** Find an angle of measure $\theta$ that is coterminal with an angle of measure $-\frac{17\pi}{6}$ where $0 \leq \theta < 2\pi$.

Let's see what other fun things we can do with angles. Given any central angle of a circle, we know that it is subtended by an arc of that circle, and we've already calculated a few arc lengths. But let's generalize that calculation. Solve the following problem:

**Problem.** A circle of radius $r$ has a central angle of measure $\theta$. What is the arc length $s$, in terms of $r$ and $\theta$, of the arc subtended by that angle?

**Exercise 7.** Use the formula you just derived to find the arc length along a circle of radius 10 of the arc that subtends an angle of $215°$.

In addition to an arc, another shape that appears when we draw a central angle over a circle is a **sector**:

**Definition** (SECTOR)**.** A sector of a circle is the two-dimensional region of the interior of the circle bounded by a central angle and the arc that subtends that angle.

Solve the following problem:

**Problem.** Given a circle of radius $r$ with a central angle of measure $\theta$, calculate the area $A$, in terms of $r$ and $\theta$, of the sector bounded by the angle and its arc. (If it helps, try to calculate the area of a specific sector, say, the sector created by a circle of radius 2 and a central angle of measure $45°$. Then, of course, generalize by stating $A$ in terms of $r$ and $\theta$.)

**Exercise 8.** Use the formula you just derived to find, given a circle of radius 20 with a central angle of measure $30^\circ$, the area of the resulting sector.

We have been imagining angles as representing rotational motion. So we might even ask a questions like, "How fast is the knob of a swinging door moving?" or, "How fast is the door swinging?"[1] These are different questions. The answer to the first will be in units length per unit time (e.g., meters per second), but the second is a questions about an angle, not a point. Just as the question "How wide is an angle?" had a ratio answer (radians), the question "How fast is an angle changing?" will have radians somewhere in the answer. We say that the answer to the first question is a **linear speed**, while the answer to the second is an **angular speed**.

**Definition** (ANGULAR SPEED)**.** As a point moves along a circle of radius $r$, its angular speed, $\omega$ (omega), is the angular rotation $\theta$ per unit time $t$:

$$\omega = \frac{\theta}{t}.$$

**Definition** (LINEAR SPEED)**.** The linear speed $v$ of a point moving along a circle of radius $r$ can be found as the distance traveled, arc length $s$, per unit time $t$:

$$v = \frac{s}{t}.$$

**Problem.** Find $v$ in terms of $r$ and $\omega$.

**Exercise 9.** Find the angular speed of a point moving in a circular motion completing one rotation every five seconds.

**Exercise 10.** Find the angular speed of a point moving in a circular motion completing 45 rotations per minute.

**Exercise 11.** A bicycle has wheels 28 inches in diameter. The wheels are rotating at 180 RPM (revolutions per minute). Find the speed at which the bicycle is traveling down the road.

**Exercise 12.** A satellite is rotating around Earth at 0.25 radians per hour at an altitude of 242 km above Earth. If the radius of Earch is 6378 kilometers, find the linear speed of the satellite in kilometers per hour.

[1] I want to emphasize emphatically that I am not under any illusions that you will ever, in your entire life, have any practical reason for measuring the speed of a swinging door. The concrete image is simply helpful in visualizing the abstract concept of an angle as rotational movement.

# More on Angles – Answer Key

*Precalculus*
*Mr. Simmons*

*Read through this handout carefully and pause to think and respond when instructed.*

Any angle can be measured as being between $0°$ and $180°$. However, imagine a door, not the door we have been picturing for the last two handouts, but a revolving door. It starts out the same as the other door, with its hinge at the origin and its knob at $(1, 0)$, but instead of swinging open $90°$, it revolves in a complete circle three and a half times, ending its rotation with its knob at $(-1, 0)$. What angle does this rotation represent?

Three full turns of a circle plus a half would have an angle measure of $3.5 \cdot 360° = 1260°$. So we say that the angle represented by the abovedescribed rotation has an angle measure of $1260°$.

But now imagine that another person, who was not there when you were pushing the revolving door, walks up and sees the door with its knob at $(-1, 0)$. What angle measure do you think he would ascribe to the angle created by the rotation of the door?

He might assume that the door only swung half a revolution, rather than three and a half revolutions. Thinking this, he would surely say that the angle measure is $180°$. While he is inaccurately measuring the angle of $1260°$, he is accurately measuring what seems to be the same angle. We say that these two angles—of $1260°$ and of $180°$—are **coterminal**, and that the angle of $180°$ is the **reference angle** of the $1260°$-angle. Study these definitions:

**Definition** (coterminal angles)**.** Two angles are said to be coterminal if and only if they have the same terminal side when in standard position.

**Definition** (reference angle)**.** An angle $t$'s reference angle is the smallest acute angle $t'$ formed by the terminal side of the angle $t$ and the horizontal axis.

Complete the following exercises:

**Exercise 1.** Find the least positive angle of measure $\theta$ that is coterminal with an angle measuring $800°$, where $0° \leq \theta < 360°$.

**Solution.** We repeatedly subtract $360°$ from $800°$ to get $440°$ and then $80°$, which is our final answer.

**Exercise 2.** Find an angle of measure $\alpha$ that is coterminal with an angle measuring $870°$, where $0° \leq \alpha < 360°$.

**Solution.** We repeatedly subtract $360°$ from $870°$ to get $510°$ and then $150°$, which is our final answer.

**Exercise 3.** Show the angle with measure $-45°$ on a circle and find a positive coterminal angle of measure $\alpha$ such that $0° \leq \alpha < 360°$.

**Solution.** The coterminal angle has measure $\alpha = 360° - 45° = 315°$.

**Exercise 4.** Find an angle of measure $\beta$ that is coterminal with an angle measuring $-300°$ such that $0° \leq \beta < 360°$.

**Solution.** The coterminal angle has measure $\beta = 360° - 300° = 60°$.

**Exercise 5.** Find an angle of measure $\beta$ that is coterminal with an angle of measure $\frac{19\pi}{4}$, where $0 \leq \beta < 2\pi$.

**Solution.** We have $\frac{19\pi}{4} - 2\pi = \frac{11\pi}{4}$ and $\frac{11\pi}{4} - 2\pi = \frac{3\pi}{4}$, so $\beta = \frac{3\pi}{4}$.

**Exercise 6.** Find an angle of measure $\theta$ that is coterminal with an angle of measure $-\frac{17\pi}{6}$ where $0 \leq \theta < 2\pi$.

**Solution.** We have $-\frac{17\pi}{6} + 2\pi = -\frac{5\pi}{6}$ and $-\frac{5\pi}{6} + 2\pi = \frac{7\pi}{6}$, so $\theta = \frac{7\pi}{6}$.

Let's see what other fun things we can do with angles. Given any central angle of a circle, we know that it is subtended by an arc of that circle, and we've already calculated a few arc lengths. But let's generalize that calculation. Solve the following problem:

**Problem.** A circle of radius $r$ has a central angle of measure $\theta$. What is the arc length $s$, in terms of $r$ and $\theta$, of the arc subtended by that angle?

**Solution.** Notice that

$$\text{arc length} : \text{circumference of circle} :: \text{angle measure} : \text{full rotation},$$

or, equivalently,

$$\frac{s}{2\pi r} = \frac{\theta}{2\pi}.$$

Solving for $s$, we get

$$s = \theta r.$$

**Exercise 7.** Use the formula you just derived to find the arc length along a circle of radius 10 of the arc that subtends an angle of $215°$.

**Solution.** The formula gives us

$$\begin{aligned} s &= \theta r \\ &= (215°)\,(10) \\ &= \left(215° \cdot \frac{\pi}{180°}\right)(10) \\ &= \frac{215\pi}{18}. \end{aligned}$$

In addition to an arc, another shape that appears when we draw a central angle over a circle is a **sector**:

**Definition** (SECTOR)**.** A sector of a circle is the two-dimensional region of the interior of the circle bounded by a central angle and the arc that subtends that angle.

Solve the following problem:

**Problem.** Given a circle of radius $r$ with a central angle of measure $\theta$, calculate the area $A$, in terms of $r$ and $\theta$, of the sector bounded by the angle and its arc. (If it helps, try to calculate the area of a specific sector, say, the sector created by a circle of radius 2 and a central angle of measure $45°$. Then, of course, generalize by stating $A$ in terms of $r$ and $\theta$.)

**Solution.** Notice that

area of sector : area of circle :: angle measure : a full rotation,

or, equivalently,

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}.$$

Solving for $A$, we get

$$A = \frac{1}{2}\theta r^2.$$

**Exercise 8.** Use the formula you just derived to find, given a circle of radius 20 with a central angle of measure $30°$, the area of the resulting sector.

**Solution.** The formula gives us

$$\begin{aligned} A &= \frac{1}{2}\theta r^2 \\ &= \frac{1}{2}(30°)(20) \\ &= \frac{1}{2}\left(\frac{\pi}{6}\right)(20) \\ &= \frac{5\pi}{3}. \end{aligned}$$

We have been imagining angles as representing rotational motion. So we might even ask a questions like, "How fast is the knob of a swinging door moving?" or, "How fast is the door swinging?"[1] These are different questions. The answer to the first will be in units length per unit time (e.g., meters per second), but the second is a questions about an angle, not a point. Just as the question "How wide is an angle?" had a ratio answer (radians), the question "How fast is an angle changing?" will have radians somewhere in the answer. We say that the answer to the first question is a **linear speed**, while the answer to the second is an **angular speed**.

**Definition** (ANGULAR SPEED)**.** As a point moves along a circle of radius $r$, its angular speed, $\omega$ (omega), is the angular rotation $\theta$ per unit time $t$:

$$\omega = \frac{\theta}{t}.$$

**Definition** (LINEAR SPEED)**.** The linear speed $v$ of a point moving along a circle of radius $r$ can be found as the distance traveled, arc length $s$, per unit time $t$:

$$v = \frac{s}{t}.$$

[1] I want to emphasize emphatically that I am not under any illusions that you will ever, in your entire life, have any practical reason for measuring the speed of a swinging door. The concrete image is simply helpful in visualizing the abstract concept of an angle as rotational movement.

**Problem.** Find $v$ in terms of $r$ and $\omega$.

**Solution.** Solving each of the above equations for $t$ gives us

$$t = \frac{\theta}{\omega} \quad \text{and} \quad t = \frac{s}{v}.$$

By substitution, then, we have

$$\frac{\theta}{\omega} = \frac{s}{v},$$

and using the formula for arc length ($s = \theta r$), we get

$$\frac{\theta}{\omega} = \frac{\theta r}{v},$$

and finally, solving for $v$,

$$v = r\omega.$$

**Exercise 9.** Find the angular speed of a point moving in a circular motion completing one rotation every five seconds.

**Solution.** The formula for angular speed gives us

$$\begin{aligned} \omega &= \frac{\theta}{t} \\ &= \frac{2\pi\,\text{rad}}{5\,\text{s}} \\ &= \frac{2\pi}{5}\ {}^{\text{rad}}\!/_{\text{s}}. \end{aligned}$$

**Exercise 10.** Find the angular speed of a point moving in a circular motion completing 45 rotations per minute.

**Solution.** The formula for angular speed gives us

$$\begin{aligned} \omega &= \frac{\theta}{t} \\ &= \frac{(45 \cdot 2\pi)\ \text{rad}}{1\,\text{min}} \\ &= 90\pi\ {}^{\text{rad}}\!/_{\text{min}}. \end{aligned}$$

**Exercise 11.** A bicycle has wheels 28 inches in diameter. The wheels are rotating at 180 RPM (revolutions per minute). Find the speed at which the bicycle is traveling down the road.

**Solution.** The angular speed is given as 180 RPM, which can be converted into

$$\begin{aligned} \omega &= \frac{\theta}{t} \\ &= \frac{(180 \cdot 2\pi)\ \text{rad}}{1\,\text{min}} \\ &= 360\pi\ {}^{\text{rad}}\!/_{\text{min}}. \end{aligned}$$

The formula for linear speed in terms of angular speed then gives us

$$\begin{aligned} v &= r\omega \\ &= (14\,\mathrm{in})\,(360\pi\,{}^{\mathrm{rad}}\!/\!_{\mathrm{min}}) \\ &= 5040\pi\,{}^{\mathrm{in}}\!/\!_{\mathrm{min}}. \end{aligned}$$

If we want to, we can convert this into a more natural unit:

$$5040\pi\,{}^{\mathrm{in}}\!/\!_{\mathrm{min}} \cdot \frac{1\,\mathrm{ft}}{12\,\mathrm{in}} \cdot \frac{1\,\mathrm{mile}}{5280\,\mathrm{ft}} \cdot \frac{60\,\mathrm{min}}{1\,\mathrm{hr}} \approx 14.99\,\mathrm{mph}.$$

**Exercise 12.** A satellite is rotating around Earth at 0.25 radians per hour at an altitude of 242 km above Earth. If the radius of Earch is 6378 kilometers, find the linear speed of the satellite in kilometers per hour.

**Solution.** The angular speed is given as $0.25\,{}^{\mathrm{rad}}\!/\!_{\mathrm{hr}}$, and the radius of the circle (not of Earth) is $6378\,\mathrm{km} + 242\,\mathrm{km} = 6620\,\mathrm{km}$, so the formula for linear speed in terms of angular speed gives us

$$\begin{aligned} v &= r\omega \\ &= (6620\,\mathrm{km})\,(0.25\,{}^{\mathrm{rad}}\!/\!_{\mathrm{hr}}) \\ &= 1655\,{}^{\mathrm{km}}\!/\!_{\mathrm{hr}}. \end{aligned}$$