GreatHearts
Irving

Remote Learning Packet

NB: Please keep all work produced this week. Details regarding how to turn in this work will be forthcoming.

April 6 - April 10, 2020
Course: 11 Precalculus
Teacher: Mr. Simmons michael.simmons@greatheartsirving.org

Weekly Plan:

Monday, April 6
- [ ] Problem set 4.7: 6-12 answer key
- [ ] "Introduction to Angles" handout

Tuesday, April 7
- [ ] Start "Radians" handout

Wednesday, April 8
- [ ] Continue "Radians" handout

Thursday, April 9
- [ ] Finish "Radians" handout

Friday, April 10
- [ ] No school!

Statement of Academic Honesty

I affirm that the work completed from the packet is mine and that I completed it independently.

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Student Signature

I affirm that, to the best of my knowledge, my child completed this work independently

____

Parent Signature

Monday, April 6

Today you will need

- This packet (preferably printed out)
- The answers for section 4.7 that are in the back of the textbook
- A pencil
- Paper
- Fortitude

1. Check your answers to problem set 4.7: 6-12.
2. Read this, here:

This concludes our discussion of exponential and logarithmic functions. We have already taken a test covering sections 4.1-4.5, and since then we have covered 4.6 and 4.7. We will be omitting section 4.8, since it deals not primarily with mathematics, but with the practical application of mathematics. A test over sections 4.6 and 4.7 is forthcoming, but due to the difficulties of testing remotely, the date of this test cannot yet be determined. So, we will move on to the next topic. When a date for the test can be set, we will then spend time in review preparing for the test, but for right now, let's put exponential and logarithmic functions to the side, and move on to another topic that will prepare us well for calculus.

Math is the study of quantity, of number and shape, and one of the fundamental aspects of shape is angle. We've all seen angles before, and we'd probably all say that we know what an angle is. But there is a difference between seeing and looking, between knowing what something is intuitively and analyzing it rigorously. Here begins our rigorous analysis of angle. The simplest shape that can be constructed from angles is a triangle, and hence the branch of mathematics into which we are about to delve is known as **trigonometry** (from Modern Latin *trigonometria*, from Greek *trigonon* "triangle" (from *tri*- "three" + *gōnia* "angle, corner" + *metron* "a measure").

Notice, we are moving on from one topic and starting another. This is a major shift. I am very sorry that we can't all be in the same room as we should be. This should be a grand transition: having routed exponential functions, we charge on to conquer the branch of mathematics perhaps most feared above all, the much-slandered *trig*. It is sad that I can't be there to encourage you in person, to quell whatever fears you might have at what may appear to be a daunting task, but we make do with what we have. Trigonometry is mathematics, and mathematics is an adventure. You are here to challenge your mind and explore the possibilities of logical deduction, to discover the beauty of these eternal truths. This will be more difficult than it should, given the isolation from each other that we are all experiencing, but again, take courage and persevere. I am available through email if you would like to reach out, and I would be happy to hear from you.

3. Read through the handout entitled "Introduction to Angles," following all instructions therein. This is mathematical writing. Mathematical reading is slow going. Read slowly. Don't move on

from a sentence until you've understood what it says. If you read at a normal pace, or even a slow but steady pace of reading aloud, you're probably moving too fast. Pause after every sentence to make sure you understand what it has said. Draw diagrams if it's helpful, even if you're not instructed to. Enjoy!

Tuesday, April 7

Today you will need

- This packet (preferably printed out)
- A pencil
- Paper
- Fortitude

1. If you have not finished reading it, finish reading yesterday's handout ("Introduction to Angles").
2. Similarly to yesterday, today you will be reading about angles. Find the handout entitled "Radians," and start reading through it, completing all instructions therein. (You will have this week to complete it, so don't worry if you don't finish by the end of 40 minutes.) Pause when it tells you to pause. Force yourself honestly to answer the questions it asks to the best of your ability without moving on until you've done so. Again, read slowly. Slooooowly. Pause after every sentence to make sure you've understood what it says. This is math. Mathematical reading is slow going. Read. Slowly.

Wednesday, April 8

Today you will need

- This packet (preferably printed out)
- A pencil
- Paper
- Fortitude

1. Continue completing the handout entitled "Radians." If you have finished, review it. Make sure you've learned all vocabulary from both handouts.

Thursday, April 9

Today you will need

- This packet (preferably printed out)
- A pencil
- Paper
- Fortitude

1. Continue completing the handout entitled "Radians." If you have finished, review it. Make sure you've learned all vocabulary from both handouts.

Introduction to Angles

Precalculus
Mr. Simmons

This handout is simply a primer on the terminology surrounding angles, so that we're all on the same page when we talk about them. Please review the following definitions and then answer the questions at the end. I hope the vocabulary here isn't too daunting.

As a preface, consider this: a good way to think about angles is in terms of rotational motion (similarly to how we talk about a ray as "emanating" or "extending" from a single point, even though it is really a fixed object that doesn't move). Imagine you're standing on the xy-plane in the fourth quadrant (bottom-right), and you want to walk into the first quadrant (upper-right), and there's a door in the way, with its hinge at the origin and its knob at $(1, 0)$. You open the door a little too fast, so the knob slams into $(0, 1)$, putting a hole through the y-axis. There, you've got a right angle. This image helps us phrase the measurement of angles in terms of rotation. Rotation "begins" when the knob is at $(1, 0)$ and "ends" when the knob is at $(0, 1)$. There's nothing actually dynamic (motion-related) about angles—since they are, like everything else in mathematics, abstract objects, eternally constant and immutable—but it's helpful to think of them as dynamic so that we finite, mutable, inconstant humans can describe and understand them more easily.

Learn these definition. You don't have to memorize them verbatim, but if asked, you should be able to provide, off the top of your head, a mathematically correct definition.

Definition (RAY)**.** A ray is a set containing one point on a line and all points extending in one direction from that point.

Definition (ANGLE)**.** An angle is the union of two rays having a common endpoint.

Definition (VERTEX)**.** A vertex is the common endpoint of two rays that form an angle.

Definition (INITIAL SIDE)**.** The initial side of an angle is the side of that angle from which rotation begins.

Definition (TERMINAL SIDE)**.** The terminal side of an angle is the side of that angle at which rotation ends.

Definition (MEASURE OF AN ANGLE)**.** The measure of an angle is the amount of rotation from the initial side to the terminal side. Conventionally, we use Greek letters as variables for the measure of an angle, typically theta (θ or ϑ), phi (ϕ or φ), alpha (α), beta (β), or gamma (γ).

Definition (DEGREE)**.** The degree ($^\circ$) is a unit of measure describing the size of an angle of one degree (1°) as one 360th of a full revolution of a circle.

Definition (STANDARD POSITION)**.** Standard position is the position of an angle having its vertex at the origin and its initial side along the positive x-axis.

Definition (POSITIVE ANGLE). A positive angle is an angle measured counterclockwise from the positive x-axis.

Definition (NEGATIVE ANGLE). A negative angle is an angle measured clockwise from the positive x-axis.

Definition (QUADRANTAL ANGLE). A quadrantal angle is an angle whose terminal side lies on an axis (e.g., an angle of $0°$, $90°$, $180°$, or $270°$).

Definition (CENTRAL ANGLE). A central angle of a circle is an angle whose vertex is the circle's center.

Complete the following exercises on a separate sheet of paper.

Exercise 1. Sketch an angle of $30°$ in standard position.

Exercise 2. Sketch an angle of $-135°$ in standard position.

Exercise 3. Sketch the angle of $240°$ in standard position as a central angle of the unit circle.

Radians

Precalculus
Mr. Simmons

Read through this handout carefully and pause to think and respond when instructed.

We got the unit called degrees (°) by dividing a full rotation into 360 equally sized angles and saying that each of those angles had measure 1°. Note that the number 360 was an arbitrary choice: we could have chosen 4, or 10, or any other counting number. But 360 is useful, because it is divisible by so many numbers (i.e., by 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, and 360 itself). Fun fact: 360 is therefore called a "highly composite" number. So are 12 and 60, which is why there are 12 inches in a foot and 60 minutes in an hour (and 60 seconds in a minute).[1]

But is there a less arbitrary way to measure an angle? When we look at an angle and wonder how big it is, we generally wonder, in an intuitive sense, how "far apart" the two rays are. A bigger angle will mean two rays that are "further apart." But, of course, the distance between the two rays is ... zero. Always. Because they're touching (at the vertex).

So what do we do? Last handout, we pictured the standard position right angle as the swinging open of a door whose hinge is at the origin and whose knob swings from $(1, 0)$ to $(0, 1)$. This dynamic representation of an angle allows us to measure the angle not by asking how "far apart" the rays are (because that number will always be zero), but by asking how far the knob has swung. So let's trace the path of the knob.

In the space below (or, if you haven't been able to print out this document, then on a separate sheet of paper), sketch the aforementioned standard position right angle, the one represented by the door swinging open, with its vertex at the origin and rays that pass through the points $(1, 0)$ and $(0, 1)$:

Now put your pencil down at $(1, 0)$ and start drawing the unit circle counterclockwise, but stop once you get to $(0, 1)$. If we think of this angle as representing the opening of the door, then what you just traced is the path of motion of the doorknob.

[1] Just for fun, consider the pros and cons of the metric system of units versus the imperial system of units. Sure, the metric system simplifies everything to base 10, but there are good reasons to use highly composite numbers like 8, 12, and 36.

What you just drew is called an **arc**. An arc may be a portion of a full circle, a full circle, or even more than a full circle. The length of the arc around an entire circle is called the **circumference** of that circle. The arc you just drew is just a portion of the unit circle (one fourth, to be specific), and it has two **endpoints**: $(1, 0)$ and $(0, 1)$.

We say that the arc you just drew **subtends** our right angle, because the angle's rays go through the arc's endpoints, and the angle's vertex is the arc's circle's center (in this case, the origin). We said earlier that measuring the path of the knob, that is, measuring this arc, would help us measure the angle. Well, this arc has length one quarter the circumference of the unit circle. Take a moment to find out exactly what that is. Write your answer here (or on a separate sheet of paper):

There, we have a number! Can we say now that that's the measure of our angle? Why or why not? Write down your thoughts:

If we look at any given angle, intuitively we want to measure it, as we've said, by the length of the arc that subtends it. But the problem is, it is subtended by many arcs. An infinite number of arcs. Take, for instance, the right angle we were just considering. Depending on which circle you choose to draw over it (centered at the origin, of course), you could make it subtended by an arc of any length you like by simply making the the circle bigger or smaller. So how can we use arc lengths to measure angles?

One way to do it is simply, as we just did, to choose the unit circle every time. Given any angle, look at the arc on the unit circle that subtends that angle, and call that arc length the measure of the angle. A right angle is subtended by an arc of length—you calculated it earlier—$\frac{\pi}{2}$.[2] So we say that a right angle has measure $\frac{\pi}{2}$. Beautiful!

But picking the unit circle still feels a bit arbitrary. What's so special about the unit circle? Instead of picking a particular circle, shouldn't we pick an arbitrary circle? Angles aren't doors, they're connected rays. Unlike doors, rays are infinitely long, and there's nothing special about one point on the ray versus another (other than the vertex, but we already said we can't use that one). Is there any way that, given an angle we're trying to measure, we can come up with a measure, a number, based on arc length, without it mattering which arc we pick? Sounds crazy. Stop and think about it. Write your thoughts here (or, as before, if appropriate, on a separate sheet of paper):

[2] This is because the unit circle has circumference $2\pi r = 2\pi(1) = 2\pi$, one fourth of which is $\frac{\pi}{2}$.

Consider that, for any given angle you're trying to measure, as you choose bigger and bigger arcs, the circles that they are portions of will also be bigger and bigger. And what does it mean for a circle to be bigger? Before moving on, take a moment to recall the precise definition of a circle.

...

A circle is the set of all points equidistant from a center point, and we call that distance the circle's radius. So a circle is defined in terms of its radius. What it *means* for a circle to be "bigger" is that its radius is longer. So the bigger the arc you choose, the bigger the radius that goes along with it. If we choose a circle of radius 2, then instead of getting an arc of length $\frac{\pi}{2}$, we get an arc of length ... well, what's a quarter of this new circle's circumference? Write it down:

That's right, the new arc length is π. That's different from $\frac{\pi}{2}$. Sounds like a problem. But wait, if we got an arc of length $\frac{\pi}{2}$ when we had a circle of radius 1, and we got an arc of length π when we had a circle of radius 2 .... Do we see a pattern? What's the pattern? Write down your thoughts:

Just as every integer, even if we don't write it as a ratio, is a ratio with a denominator of 1, so is every angle measure a ratio. When we pick the unit circle, we are choosing a denominator of 1; when we pick a circle of radius 2, we are choosing a denominator of 2; when we pick a circle of radius 3, we are choosing a denominator of 3; and so on. (I'm picking integers only because they're simple—you could pick literally any positive real number.) But

$$\frac{\frac{\pi}{2}\text{ arc length units}}{1\text{ radius unit}} = \frac{\pi\text{ arc length units}}{2\text{ radius units}} = \frac{\frac{3\pi}{2}\text{ arc length units}}{3\text{ radius units}} = \cdots = \frac{\pi}{2}.$$

So we can reasonably say, without any arbitrary choice, that the measure of a right angle is $\frac{\pi}{2}$. The ratio of arc length to radius doesn't change depending on which circle we pick. Given any angle, if you take an arc that subtends that angle and divide its length by its circle's radius, no matter which circle you choose, you always get the same answer. *The ratio of arc length to radius is constant for any given angle.* Sounds like we've found ourselves a consistent way to measure angles using arc lengths! This is particularly satisfying because it fits with our intuitive notion of angles as representing rotation. This way of measuring angles tells us quite straightforwardly how far the knob of our door has traveled, which is an intuitive way of picturing the size of the angle. Wonderful.

This measure of an angle—the one we get by dividing the arc length (of an arc that subtends the angle) by the radius of the circle (of which that arc is a portion)—is called the **radian measure** of an angle.

Teachnically speaking, the radian measure of an angle is stated in a unit called **radians**, where one radian is defined as the measure of the angle subtended by an arc on the unit circle of arc length 1—but very rarely does any mathematician write out the word "radians" or even the abbreviation "rad" next to the radian measure of an angle, and very rarely is that angle I just described, the one whose measure is 1 radian, ever used for anything. (It's also kind of ugly.[3]) Since the radian measure of an angle is gotten by dividing a length (an arc length) by another length (a radius), the length units cancel, leaving radians a dimensionless unit, or what mathematicians call a "pure number."

[3] Fun exercise: explain why an angle of radian measure 1 is ugly. Or, alternatively, argue that it is beautiful. Feel free to email me with responses.

That was a lot of work! As a way of solidifying the concepts covered in the preceding pages, go ahead and read through the rigorous statements of the definitions you just learned. While learning these definitions verbatim is not necessary, you should be able to give a complete, mathematically precise definition of each of these words from memory.

Definition (ARC)**.** An arc is a portion of a circle.

Definition (SUBTEND)**.** An arc subtends an angle if and only if the angle's two rays pass through the arc's two endpoints.

Definition (RADIAN MEASURE)**.** The radian measure of an angle is the ratio of the length of the arc that subtends the angle to the radius of the circle.

In other words, if s is the length of an arc of a circle, and r is the radius of the circle, then the central angle containing that arc measures $\frac{s}{r}$ radians. In a circle of radius 1, the radian measure corresponds to the length of the arc.

Definition (RADIAN)**.** One radian is the measure of the central angle of a circle such that the length of the arc between the initial side and the terminal side is equal to the radius of the circle.

Complete the following exercises on a separate sheet of paper.

Exercise 1. Find the radian measure of one third of a full rotation.

Exercise 2. Find the radian measure of three fourths of a full rotation.

Exercise 3. Remember that a conversion factor is a fraction, equal to one, that you multiply a measurement by to change its units. For example, to change 2 feet into inches, I multiply 2 feet by the conversion factor $\frac{12 \text{ in}}{1 \text{ ft}}$ to get

$$2 \text{ ft} \times \frac{12 \text{ in}}{1 \text{ ft}} = 24 \text{ in}.$$

The feet cancel, leaving only inches.

Come up with conversion factors to convert from degrees to radians and from radians to degrees.

Exercise 4. Convert the radian measure $\frac{\pi}{6}$ into degrees.

Exercise 5. Convert the radian measure 3π into degrees.

Exercise 6. Convert $15°$ into radians.

Exercise 7. Convert $126°$ ito radians.

Exercise 8. In a clear, neat diagram, draw the unit circle, and then sketch in standard position the following angles, given in portions of a full rotation. Then label, at the intersection of each angle's terminal side with the unit circle, the measure of that angle in both degrees and radians. (You may include the unit label "radians" or "rad" on the radian measure if you would like, but you needn't.)

0, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{6}$, $\frac{5}{6}$, $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, $\frac{7}{8}$, $\frac{1}{12}$, $\frac{5}{12}$, $\frac{7}{12}$, and $\frac{11}{12}$

For example, to draw and label an angle that is $\frac{1}{4}$ of a full rotation, you would draw the standard position right angle that we were dealing with all throughout this handout (the one represented by the swinging door) and label it, at the point $(0, 1)$, with the labels "$90°$" and "$\frac{\pi}{2}$."