GreatHearts Irving

Remote Learning Packet

April 27-May 1, 2020
Course: 11 Precalculus
Teacher(s): Mr. Simmons

Weekly Plan:

Monday, May 4
- ☐ Story time!
- ☐ Read "Right triangles"
- ☐ Problems 1-5 odd

Tuesday, May 5
- ☐ Read "The Sine, Cosine, and Tangent functions"

Wednesday, May 6
- ☐ Problems 1-5 (all)

Thursday, May 7
- ☐ Read "Relationships between Trig Functions"

Friday, May 8
- ☐ Problems 1-8 (all)

Monday, May 4

1. Story time! If technologically feasible, send me an email telling me a fun story that happened recently. I miss you! (yes, you)

Learning from packets is much harder than learning in class. I've found (after much searching) a good resource for learning trigonometry. We'll be using it for the remainder of the year. It's the second part of a book called *Precalculus* by Mr. Joseph Gerth. I've included it as a material on Google Classroom.

2. Read "Right triangles" from Gerth's *Precalculus* (pp. 84-90).
3. Complete problems 1, 3, and 5 (pp. 90-91)

Tuesday, May 5

1. Read "The Sine, Cosine, and Tangent functions" (pp. 93-109).

Wednesday, May 6

1. Complete problems 1-5 (pp. 109-110) on the same piece of paper from Monday.

Thursday, May 7

1. Read "Relationships between the Trig functions" (pp. 113-121)

Friday, May 8

1. Complete problems 1-8 (pp. 121-122) on the same piece of paper from Monday and Wednesday.

Part two: Trigonometry

Trigonometry is the study of angles and their relationship to triangles. As you'll soon see, we attempt to relate the angles of a triangle to its sides. It is perhaps unsurprising that this Part is closely related to Geometry – we'll ask that famous question, "What is the relationship?" What is interesting, however, is that this Part is also closely related to Algebra. We will often work with equations and use algebra to change them into something different.

This reveals one reason why Trigonometry is studied: It is, in some sense, a union of those two great disciplines of Geometry and Algebra. It takes what you know in each discipline and applies it, so that you can expand each.

This branch of mathematics is also studied at this point because it is a very important Calculus concept. Calculus teachers and textbooks will assume that you can evaluate, e.g., $\sin\frac{\pi}{4}$ in seconds. You will also be expected to have deep insight into the graphs of the Trigonometric functions. Since you'll often be finding the area of a Trigonometric function or, perhaps, determining its behavior as it goes to infinity.

Many students struggle with Trigonometry. This is mostly likely due to their deficiencies in Geometry, Algebra, or both. We will give a small review of right triangle Geometry, which we think will be helpful for you. But many of you may need to do some reading, research, and practice on your own to get you up to par. Even when you are well-versed in the pre-requisites, you must also be prepared to spend time practicing and mastering new techniques as well.

Another issue that you must be aware of is that Trigonometry is very vertical. That is, anything learned serves as a foundation for the next thing. If you don't learn something, therefore, it's nearly impossible to progress. The cornerstone of this Part is most assuredly right triangles. You simply must master the special right triangles presented in the first section. Failure to do so will nearly guarantee failure. As such, you must approach each section with due diligence, and master any and all of the content.

Unit four

Right triangle Trigonometry

Τριγωνομετρία – Greek for Trigonometry. Literally meaning "triangle measuring."

Trigonometry is a branch of mathematics that looks at triangles, angles, and circles. It's a complex and often rigorous art that requires intuition over skills. It is imperative to develop and master the foundational knowledge of Trigonometry, because without it, success is very difficult.

Although we assume the reader knows nothing of Trigonometry, it might be wise to have some experience before beginning this Unit. Additionally, there are a few Geometry theorems that are assumed to be known. If any of the following seems difficult, it is highly recommended to consult the previous texts.

We begin by studying some basic Geometry. After this, we then introduce the basic Trigonometric functions: Sine, cosine, and tangent.

§1 Right triangles

Since the word "Trigonometry" literally means "triangle measure", perhaps it is no surprise that we begin working with triangles. Recall that a **triangle** is a **polygon** with three sides, and that a right triangle is so called because one of its angles measures 90° (which we define as a **right angle**). Thus, Figure 21 is a right triangle.

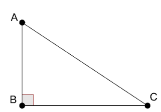


Figure 21

All right angles will have a box (like the one next to B) to denote the fact that they are 90°. Recall that it is not good enough to "look like" it's 90° - it *must* have the box.

This simple object contains many profound truths, some of which you may have learned in Geometry class. Perhaps the most well-known and profound truth is known as Pythagoras' Theorem. It has been known and used for millennia, contains more than 300 different proofs, and is used in just about every branch of mathematics.[i]

Pythagoras' Theorem

Given a right triangle with lengths a, b, c, where c is the **hypotenuse** and a, b are the **legs**, then

$$a^2 + b^2 = c^2.$$

[i] Even statistics has a use for Pythagoras' Theorem!

We will usually prove each basic and foundational assertion in this course. However, we believe that most will have already proved this theorem on their own. Therefore we will not prove this assertion here.[ii]

Example 1

Find the length of the missing side in Figure 22.

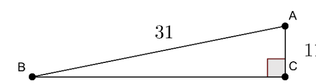


Figure 22

Pictures are not (and never will be) drawn perfectly to scale.

The hypotenuse – always opposite of the right angle – is side length $\overline{AB}$, which has a length of 31. Thus we can say that c, which represents the length of the hypotenuse in Pythagoras' Theorem, is 31. And since $\overline{AC}$ is one of our legs, we will let

$$a = 11.$$

Given this, we now have

$$11^2 + b^2 = 31^2$$

$$121 + b^2 = 961.$$

This is now a simple algebra problem. Solving for b we get

$$b^2 = 840$$

$$b = 28.98.$$

Some right triangles are special; they come in knowable ratios. And if we know the ratios of all three sides, then we do not need to use Pythagoras' Theorem at all.

Consider the right triangle shown in Figure 23.

[ii] The internet is a great resource. Use it.

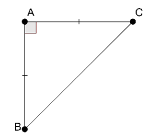
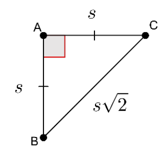


Figure 23

The tick marks on the two legs indicate that those two lines are the same length.

This is an **isosceles triangle**, which means that two of the side lengths are **congruent**. Consequently, we can make the following assertion.

Isosceles right triangle side ratio



Given an isosceles right triangle with a leg of length s, then the other leg has a length of s and the hypotenuse has a length of $s\sqrt{2}$.

Proof.

Construct isosceles right triangle ABC with leg $AB = s$. Then $\overline{AC} = s$ because we have an isosceles right triangle.

Further, we know that the length of the hypotenuse is

$$\overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2$$

by Pythagoras' Theorem. But we know the length of both $\overline{AB}$ and $\overline{AC}$, and so, by substitution, we have

$$s^2 + s^2 = \overline{BC}^2.$$

Whence, after simplification, we have

$$2s^2 = \overline{BC}^2$$

$$\overline{BC} = s\sqrt{2},$$

Which is what we wanted to show.

Example 2

Determine the lengths of the missing sides in Figure 24.

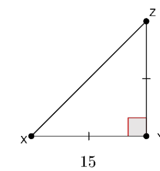


Figure 24

Applying the special triangle ratio, we have that $s = 15$. Thus the other leg,

$$\overline{YZ} = 15,$$

and the hypotenuse,

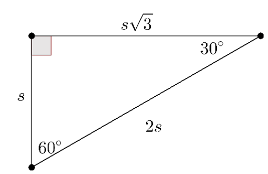
$$\overline{XZ} = 15\sqrt{2}.$$

Keep in mind Pythagoras's Theorem can still be used to determine the previous triangle, although you would have to apply the fact that $\overline{XY} = \overline{YZ}$.

Let's take a look at one more special right triangle ratio.

$30° - 60° - 90°$ Right triangle ratio

Given a right triangle with angles of $30°, 60°$, and $90°$, then the short leg has length s, the longer leg has a length of $s\sqrt{3}$, and the hypotenuse has a length of $2s$.



We will prove the previous result, but our proof will be a bit messy. We'll look for a better way to prove it as we continue through this Unit.

Proof.

To fully prove this assertion, we will need to prove three different possibilities:

I. A right triangle with legs of s and $s\sqrt{3}$,
II. A right triangle with a leg of s and a hypotenuse of $2s$, and
III. A right triangle with a leg of $s\sqrt{3}$ and a hypotenuse of $2s$.

We will prove the first point, then leave the final two proofs to the reader.

We are given a right triangle with lengths of s and $s\sqrt{3}$. Thus, using Pythagoras, we have

$$a = s, b = s\sqrt{3}$$

whence

$$s^2 + \left(s\sqrt{3}\right)^2 = c^2.$$

Consequently, we get that

$$s^2 + 3s^2 = c^2$$
$$4s^2 = c^2$$
$$c = 2s,$$

which is what we wanted to show.

Example 3

Complete the right triangle shown in Figure 25.

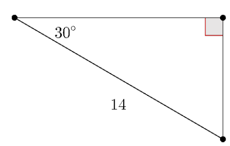


Figure 25

To "complete" a right triangle means to determine all of its side lengths and angles measures. In our case, we must determine one angle measure and the length of both legs. We begin with the angle measure. Recall from Geometry that the sum of three angles in a triangle is always 180°. Thus our missing angle (which we'll call α)[iii] is

$$\alpha = 180° - (30° - 90°)$$

[iii] It is my custom to name sides with lower-case Latin letters, a, b, c, and then call their opposite angles with lower-case Greek letters, α, β, γ, respectively. This is my own personal convention, and I welcome you to develop your own.

$$\alpha = 60°.$$

We'll use the ratios shown in our previous result to determine the side lengths. Recall from Geometry that that the shortest side of a triangle will always be opposite the shortest angle. Since, in a $30° - 60° - 90°$ triangle, the hypotenuse is always twice the length of the shortest side (which we'll call b), we can write

$$14 = 2b,$$

And hence

$$b = 7.$$

To determine the one remaining side, which is the longer leg (since it is opposite the 60° angle), we simply multiply our previous result by $\sqrt{3}$. Hence our remaining side (which we'll call a) is

$$a = 7\sqrt{3}.$$

We show our final results in Figure 26.

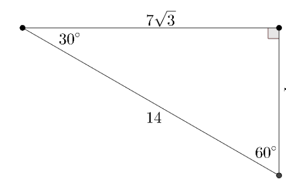


Figure 26

Pythagoras' Theorem allows us to determine the third side of a right triangle when only given two other sides. This is very useful, but it leaves out some other information. For example, what if we wanted to determine the angle measures when given two side lengths? Unless we measure it, we wouldn't be able to tell.[iv] Or, what if we had only known one of the sides of our triangle? Then we could not determine anything else.

This is one of the reasons that these special right triangles are so useful. We'll use that fact a myriad of times in this Part – and we'll come back to it very shortly. Doubtless, of course, you also see the advantage to solving a right triangle as quickly and efficiently as we just demonstrated.

[iv] And even if we did measure it, we would have to draw our triangle perfectly to scale, and even then, there is a degree of error in measurement.

Allow us to reiterate the importance of this section: You simply must become very adept at solving special right triangles. We cannot overemphasize how foundational this is. You will utilize this tool again and again in the forthcoming sections. We've given you many practice problems in this section, but you will need to take extra care to master it for yourself.

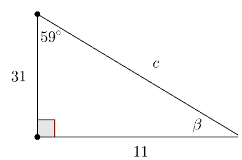
§1 Exercises

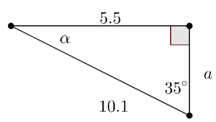
1.) In the following problems, assume that a and b are the lengths of legs, and that c is the length of a hypotenuse. Find the length of the missing side.
   - (A) $a = 12, b = 15$
   - (B) $a = 1.5, b = 2.3$
   - (C) $a = 24, b = 8$
   - (D) $a = 12.5, c = 19$
   - (E) $b = 113, c = 201$
   - (F) $c = 4.5, a = 1$
   - (G) $c = 27, b = 24$
   - (H) $a = 10, b = 1$

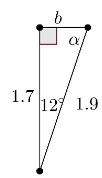
2.) Assume you have a triangle with leg lengths of 5 and 7.
   - (A) Troy lets $a = 5$ and $b = 7$. Tina lets $a = 7$ and $b = 5$. Do they get the same result for c?
   - (B) Prove that a and b are interchangeable when using Pythagoras' Theorem.

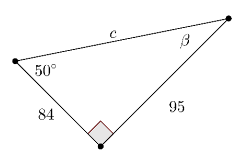
3.) Are all triangles possible? For example, can you have a right triangle with legs of length 10 and 20 and a hypotenuse of 15? Why or why not?

4.) Recall that all triangles' angles add up to 180°. With this in mind, complete the following right triangles.

(A)


(B)


(C)


(D)


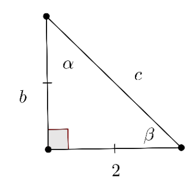
5.) So far, we've seen that $a^2 + b^2 = c^2$ is true for all real numbers – that is, at least one of the three lengths was irrational. Is it possible that all three lengths are integers? Let's explore…
   - (A) $a = 3, b = 4, c = ?$
   - (B) $a = ?, b = 12, c = 13$
   - (C) See if you can discover three other **Pythagorean Triples**.

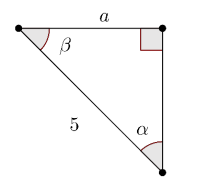
6.) The following table represents a small list of Pythagorean Triples. (Each row contains the three side lengths of the triangle.)

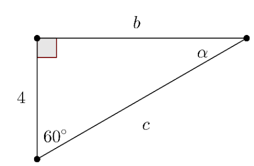
Side lengths	3	4	5
Side lengths	6	8	10
Side lengths	9	12	15

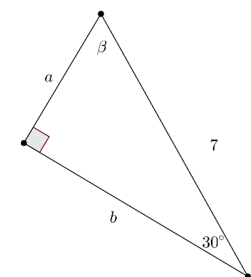
   - (A) What pattern exists in this table?
   - (B) Use the pattern to list out three more Pythagorean Triples.
   - (C) How many Pythagorean Triples exist?
   - (D) These were all multiples of the smallest (and most common) Pythagorean Triple, the 3,4,5 triangle.[v] Will this trick work with other distinct Pythagorean Triples, like the 5,12,13 triangle?

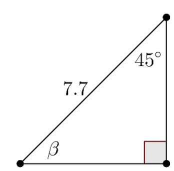
7.) Complete the following right triangles.

(A)


(B)


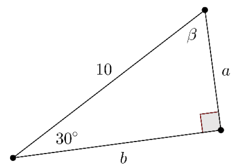
(C)


(F)


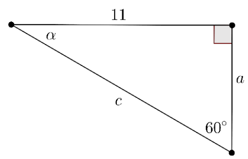
(G)


[v] We highly encourage you to memorize the 3,4,5 triangle. It pops up enough that memorizing it will save you some considerable time.

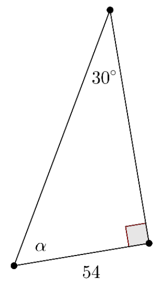
(D)



(E)



(H)



8.) As a matter of convention, we often rewrite any ratios that have an irrational number in their denominator such that the numerator is entirely rational. For example,

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

In most cases, this is purely by convention. One reason for doing this is that you will only need to memorize one set of numbers when it comes to specific Trig ratios.

(A) What is the denominator in the example shown above? And what is the numerator and denominator of the number multiplied by $\frac{1}{\sqrt{2}}$?

(B) What is $\frac{\sqrt{2}}{\sqrt{2}}$?

(C) Based on your answer to (B), has the value of $\frac{1}{\sqrt{2}}$ changed after the multiplication? Use a calculator if you're not convinced.

(D) Use the above example to rationalize the following denominators.

i. $\frac{3}{\sqrt{3}}$ — v. $\frac{3}{\sqrt{4}}$

ii. $\frac{4}{\sqrt{2}}$ — vi. $\frac{5}{2\sqrt{2}}$

iii. $\frac{\sqrt{2}}{\sqrt{3}}$ — vii. $\frac{24}{\sqrt{12}}$

iv. $\frac{\pi}{\sqrt{7}}$ — viii. $\frac{\sqrt{3}}{\sqrt{10}}$

9.) Answer True or False.

(A) A $45° - 45° - 90°$ triangle can never have three side lengths which are all whole numbers.

(B) The shortest side of a triangle is always opposite of the shortest angle.

(C) The hypotenuse is always the longest side of a triangle.

(D) A triangle can never have more than one 90° angle.

(E) There is an infinite amount of Pythagorean Triples.

10.) In your own words, describe the shortcut to completing an isosceles right triangle.

11.) In your own words, describe the shortcut to completing a $30° - 60° - 90°$ triangle.

12.) Will you forget these special triangle ratios?

13.) Where will you look if you forget them?

14.) You're sure you won't forget them??

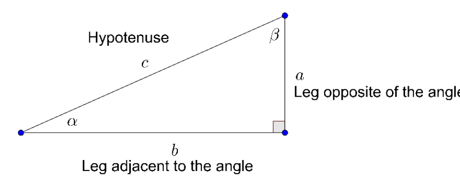
§2 The Sine, Cosine, and Tangent functions

We used Geometry in the previous section (and we will frequently fall back on it) and it had some good uses, but in this section we will discover a new set of tools that will allow us to do even more.

More importantly, though, is that we'll be able to relate angle measures and side lengths. Certainly, this will allow us to calculate more than we could previously, but we will also find a shred of truth that we felt was there but couldn't quite ascertain.

To begin with, we'll define a function that takes an acute angle as its input and returns a ratio of two side lengths.

The Sine function



The Sine function accepts as an input an acute angle of a right triangle, and then returns the ratio of the side opposite to the angle to the hypotenuse. Symbolically,

$$\sin\alpha = \frac{a}{c}.$$

We could also state that

$$\sin\beta = \frac{b}{c}.$$

Note that our current definition only accepts acute angles of a right triangle.

Example 1a

Write the ratio of the sine of angle α and β given Figure 27.

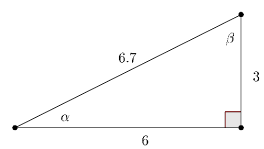


Figure 27

Let's start with the sine of angle α, which is denoted as $\sin\alpha$. This is just a ratio – so all we need to do is create a fraction. Since we've defined this function as the side opposite to the hypotenuse, we just need to write

$$\sin\alpha = \frac{3}{6.7},$$

Since 3 is the side opposite and 6.7 is the hypotenuse. It really is that simple!

The sine of angle β is handled the same way, except notice that the side opposite β is different than the side opposite of α. In this case we have

$$\sin\beta = \frac{6}{6.7}.$$

Whatever angle α is, when we input it into the sine function, we receive as an output $\frac{3}{6.7}$. The same is true for β; when we input β into the sine function, we receive as an output $\frac{6}{6.7}$.

Of course, that hasn't really revealed to us any new information. Let's try a problem that teaches us something, shall we?

Example 1b

Evaluate $\sin 30°$ given Figure 28.

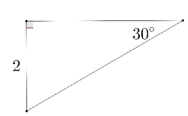


Figure 28

To answer this, we simply need to set up our ratio. Based on what we've been given, we have

$$\sin 30° = \frac{2}{\text{hypotenuse}}.$$

But what is the length of the hypotenuse? We can easily find that out using what we learned in the previous section.[i] Since 2 is the length of the short leg, then the hypotenuse must be twice that length. Hence we conclude that they hypotenuse has a length of 4. Therefore

$$\sin 30° = \frac{2}{4} = \frac{1}{2}.$$

So what did we find out? Well, if you input 30° into the sine function, you should get out $\frac{1}{2}$. But there's a very important question to answer: Will this only happen when the short leg of a $30° - 60° - 90°$ triangle has a length of 2? Put another way, does the size of the triangle play a role in determining the result of $\sin 30°$?

To answer this question, we will do as all mathematicians do: We will explore, we will experiment, we will play! We'll make a new triangle with an angle of 30° that has a different size and observe what happens.

Example 1c

Evaluate $\sin 30°$ using Figure 29.

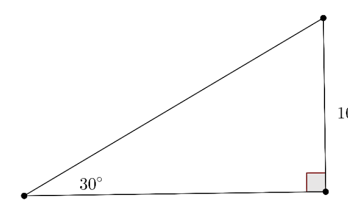


Figure 29

[i] This is exactly what we mean we say that you will need to be very adept at the previous section. This is often how we'll calculate our Trig functions – by setting up a special right triangle, and filling in the rest of the information. Therefore, we'll state it again: Learn the previous section!

We apply the same procedure. In this case, the hypotenuse must be 20. Since the sine ratio is $\frac{\text{opposite}}{\text{hypotenuse}}$, and the side length opposite to $30°$ is 10, we conclude that

$$\sin 30° = \frac{10}{20} = \frac{1}{2}.$$

Now wait just a minute, that's very interesting! Why did a different size triangle give us the same result? This is most unexpected!

Indeed, much of mathematics was discovered in the same way. It starts with a simple curiosity, and then a conscious effort to poke and prod whatever you have until it gives up some profound truth.[ii] Think of math like a piñata, which contains so much delicious truth. But the only way to get at it is to swing at it a few times (and oftentimes this swinging is truly difficult!).

In this case, the profound truth seems to be that $\sin 30° = \frac{1}{2}$ *no matter what size triangle we have*. Thus we share the following very important theorem.

The dependency of $\sin\alpha$
The output of $\sin\alpha$ only depends on α. The size of the triangle is irrelevant.

There is, of course, a reason that $\sin\alpha$ does not depend on the size of the triangle, and we'll briefly explore this in the Exercises.

This profound truth will allow us to construct *any* size triangle we wish when we go about determining the sine of any angle. This is useful, since some choices will be easier than others. In fact, it is this very fact that will motivate our work in the next unit. Let's briefly explore this idea.

Example 1d

Evaluate $\sin 45°$.

Based on what we just learned, we should create a $45° - 45° - 90°$ triangle, and then create a ratio using the lengths of the opposite side and the hypotenuse. It would be fair to ask what triangle, exactly, you should create. We start with the basics in Figure 30.

[ii] And this process of experimentation and verification can be as quick and painless as we just witnessed, or it could take several years. Or centuries. But it's the journey that counts, right?

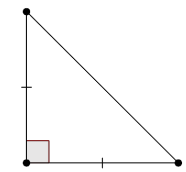


Figure 30

We've drawn some isosceles right triangle. Now how should we label the angles and the side lengths? Certainly, the two non-right angles must be $45°$ (since they are both equal and must equal $90°$), but what about the side lengths? We could, for example, start with the legs, and label each, say, 10. Once we've made that decision, we must conclude that the hypotenuse is $10\sqrt{2}$.[iii] We draw this in Figure 31.

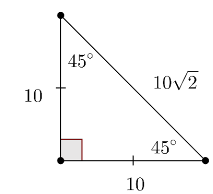


Figure 31

There's nothing *wrong* with the previous picture, but it also isn't the *best*. Should we choose a small number? Or, shouldn't we choose some number where we won't have any simplification?

Because of this, why don't we let the legs be of length 1.[iv] Then we have Figure 32.

[iii] See the previous section.
[iv] You may not see that this is the "simplest" number. That's fine! This is something you find out through experimentation and exploration.

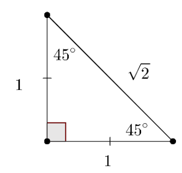


Figure 32

To calculate $\sin 45°$, then, we just need to set up our ratio accordingly.[v] We have

$$\sin 45° = \frac{1}{\sqrt{2}}.$$

What if we had chosen the original triangle, where the legs had a length of 10? Then we would have had

$$\sin 45° = \frac{10}{10\sqrt{2}} = \frac{1}{\sqrt{2}},$$

Which is the same answer, although it was one less step to get to the answer. In future sections, we'll be picking a very particular triangle to work with when we want to evaluate some Trig function. We are just being clever, and trying to save ourselves some time – although this is not mandatory, and you are free to pick any triangle you like.[vi]

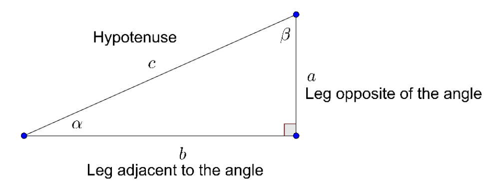
A single step may seem insignificant, but since we'll be going back to these two special triangles again and again,[vii] you'll want to streamline the process as much as possible. Regardless, we now have another value or ratio to add to our list, which you'll work on completing in the Exercises.

One more note before we continue on: Most textbooks will tell you that $\sin 45° = \frac{\sqrt{2}}{2}$. Are they wrong? And where did they get that answer from? And why would they bother writing it like that? Good questions!

Let us now move on to the Cosine function. By its very name, we can infer that it is related to the sine function, and – as we'll soon see – this is very much the case.

[v] Will it matter which 45° angle we choose? See for yourself!
[vi] We will reiterate this point in future sections.
[vii] And again!

The Cosine function



The Cosine function accepts as an input an acute angle of a right triangle, and then returns the ratio of the side adjacent to the angle to the hypotenuse. Symbolically,

$$\cos\alpha = \frac{b}{c}.$$

We could also state that

$$\cos\beta = \frac{a}{c}.$$

The Cosine function is very similar to the Sine function. It, too, accepts an angle as an input. But this time it returns as an output the side length adjacent[viii] to α over the hypotenuse. This – again – has only been defined for acute angles in a right triangle.

Example 2a

Write the Cosine ratio of α given Figure 33.

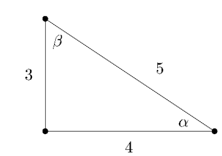


Figure 33

All we need to do is make a fraction of the length of the adjacent leg over the hypotenuse. This is simple: We get

[viii] Note that adjacent means touching. And while, yes, the hypotenuse is touching α, what we really mean is the adjacent leg.

$$\cos\alpha = \frac{4}{5}.$$

It is important to note that these ratios will *only* work with right triangles. Was the above triangle a right triangle? It might *look* that way, but how can we be sure?[ix] Up to this point, we've always seen the box at the right angle. Here we don't have one – so is our result incorrect?

This is an excellent question and one you must be aware of. Pythagoras' Theorem tells us that for any right triangle,

$$a^2 + b^2 = c^2.$$

The converse of Pythagoras' Theorem – that is, if $a^2 + b^2 = c^2$ then you have a right triangle – is also true. Since

$$3^2 + 4^2 = 5^2,$$

we can conclude we have a right triangle. Thus the aforementioned conclusion (that $\cos\alpha = \frac{4}{5}$) is correct.

Example 2b

List out the Sine and Cosine ratios for angles α and β in Figure 34.

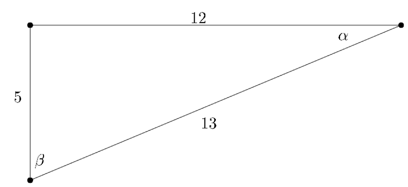


Figure 34

Remember to first verify that you have a right triangle. Since

$$5^2 + 12^2 = 13^2,$$

we can move on to the next step.

[ix] For example, if the unnamed angle, call it γ, is equal to 89.99 (which is really close to 90) then our Trig ratios are inaccurate. We must be positive we have an actual right triangle; it is not good enough to *almost* be a right triangle.

Let us deal with α first. Our ratios for α are

$$\sin\alpha = \frac{5}{13}$$

and

$$\cos\alpha = \frac{12}{13}.$$

Then, with respect to β, we have

$$\sin\beta = \frac{12}{13}$$

and

$$\cos\beta = \frac{5}{13}.$$

But isn't there something peculiar about our results? Look again – is there anything that you can see?

Why is it that

$$\sin\alpha = \frac{5}{13} = \cos\beta\,?$$

Or that

$$\sin\beta = \frac{12}{13} = \cos\alpha\,?$$

This is a note-worthy find! Why should two different functions (with different inputs) produce the same output?

Before we formalize our next theorem, let us consider the relationship that exists between the two non-right angles in a right triangle.

First, let us (in Figure 35) draw some right triangle.

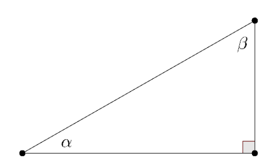


Figure 35

Let us label the two non-right angles as α and β. Is there some relationship between them?

We now need some sort of starting point – so let's think back to other relationships in triangles. The biggest one, perhaps, is the theorem that the sum of all three angles in a triangle are $180°$. So let's begin with this, play with it a bit, and then see if we can't uncover something of note.[x]

Using the previous relationship, we would then have

$$\alpha + \beta + \gamma = 180.$$

But we know $\gamma = 90°$, since it is a right angle (and by definition, all right angles equal $90°$). Substituting this in, we see that

$$\alpha + \beta = 90°.$$

This yields us one important relationship: α and β are complementary. This seems benign… But let us consider the word *Cosine* for a moment. It is actually a shortened form of the term "Complement of the sine." Wait a moment… Is it a coincidence that we just showed that α and β are complementary and the Cosine function has in its very name the word "complement"? Most certainly not![xi] We've discovered some profound truth! We now provide a formal proof.

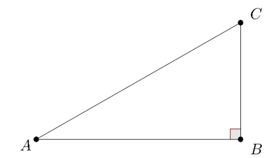
Proof.

We want to show that the two non-right angles of a right triangle must be complementary, that is, that their sum must be $90°$.

Construct right ΔABC with right angle B.

[x] Of course, our exploration may turn out fruitless. That's OK! Believe it or not, finding out what doesn't work can also very helpful. Additionally, don't get discouraged if you're not correct the first time – very few mathematicians are!

[xi] Which doesn't mean "Never". It just means unlikely. Some may scoff at the simple logic we used here, but keep in mind we're showing a process of coming to an informal conclusion. You may wish the explanation was more formal, and if that's the case, I commend you and encourage you to write your own textbook. ☺



Because the sum of all three angles in a triangle is $180°$, we know that $A + B + C = 180°$. But $B = 90°$, since (by definition) all right angles are equal to $90°$. Then, by substitution, we have that $A + 90° + C = 180°$. Using the subtraction property of equality reveals that $A + C = 90°$, which is precisely what the word "complementary" means.

We have thus shown what we wanted to show.

With this in mind, let us formalize the previous.

The relationship between the Sine and Cosine functions. The two non-right angles in a right triangle are always complementary.

Example 2c

Evaluate $\cos 30°$.

We go through the same procedure found in Examples 1c and 1d. We first draw a right triangle[xii] (Figure 36).

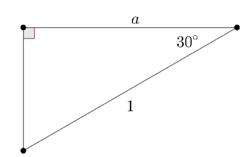


Figure 36

We chose to make our hypotenuse 1 here. There are a few other nice choices, as well.

Based on the triangle that we chose, we have that

$$\cos\alpha = \frac{\text{adjacent}}{1}.$$

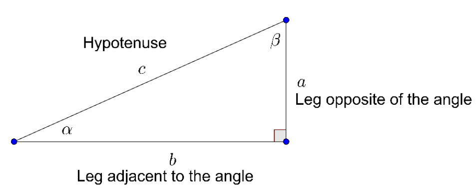
[xii] Remember that we are free to draw any right triangle we want. Well, as long as is it has a $30°$ angle, anyway.

But what is the length of the leg adjacent to α? We can figure this out because this is a special right triangle, viz., it is a $30° - 60° - 90°$ triangle. Since our hypotenuse is 1, the short leg (not marked) must measure $\frac{1}{2}$. Then, to find the long leg, we multiply $\frac{1}{2}$ by $\sqrt{3}$. Thus the long leg, a, must be $\frac{\sqrt{3}}{2}$. Since $a = \frac{\sqrt{3}}{2}$, we conclude that

$$\cos\alpha = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}.$$

We have one more Trigonometric function to uncover. We'll define it here, but in the exercises you will *derive*[xiii] it yourself.

The Tangent function



The Tangent function accepts as an input an acute angle, and then returns the ratio of the side opposite to the angle to the side adjacent to the angle. Symbolically,

$$\tan\alpha = \frac{a}{b}.$$

We could also state that

$$\tan\beta = \frac{b}{a}.$$

The Tangent function appears a bit different from the Sine and Cosine function, and, to be sure, there are some key differences. But you'll also discover in the exercises that it is closely related to the Sine and Cosine functions! Additionally – as you'll soon see – we work with it in the same manner.

Example 3a

Write the Sine, Cosine, and Tangent ratios using α of the triangle shown in Figure 37.

[xiii] What this means is that you'll take something you already know, work with it, and come out with the function we're about to define! Obviously this is the better way to do it, but we don't want to spoil your fun.

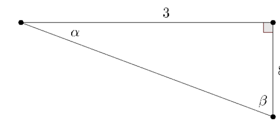


Figure 37

All we need to do is substitute. The only problem we have is that we don't have a length for the hypotenuse. No matter – this is easily found. We use Pythagoras' Theorem and find that the hypotenuse must have a length of 8.54. With this in mind, we have

$$\sin\alpha = \frac{8}{8.54},$$

$$\cos\alpha = \frac{3}{8.4},$$

and

$$\tan\alpha = \frac{8}{3}.$$

Example 3b

Evaluate $\tan 45°$.

We follow the same procedure as Examples 2c, 1c, and 1d. We create any size right triangle with a $45°$ angle, as in Figure 38.

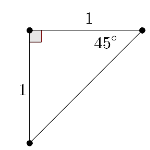


Figure 38

Recall that any $45° - 45° - 90°$ triangle will also be isosceles. Both legs, therefore, are also congruent.

Now all we need to do is write our ratio. We have

$$\tan 45° = \frac{1}{1} = 1.$$

Thus we conclude that $\tan 45°$ is 1.

Usually, evaluating a Trig function given any input is very difficult to do. Of course, in today's world, we have ready access to calculators, which are very good at the tedious process of approximating, so it isn't too difficult to evaluate $\sin 2°$, for example. You'll need to have a calculator on hand.

Example 4a

Determine the length of a given Figure 39.

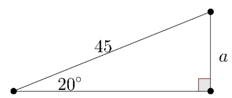


Figure 39

As we did extensively in Geometry, we should first establish some relationship. For example, we know (from Pythagoras) that $a^2 + b^2 = c^2$. In this case, however, we don't know a or b, and so Pythagoras' Theorem yields us no help. Since we have been given an angle measure and one side length, it stands to reason that we should use a Trig function. We have three different choices (Sine, Cosine, and Tangent), so which one should we use? In this example, we will try all of them and show you that there is really only one good choice.[xiv] First, let us try the Sine function. Given that $\sin \alpha = \frac{\text{opposite}}{\text{hypotenuse}}$, we have (after substitution)

$$\sin 20° = \frac{a}{45}.$$

If we try the Cosine function, we have

$$\cos 20° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{?}{45},$$

where the question mark represents the unknown length of the adjacent.

If we try the Tangent function, we would have

$$\tan 20° = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{?},$$

where, again, the question mark represents the same unknown length of the adjacent.

[xiv] This won't always be the case. Sometimes there will be two or three good choices.

Based on what we just saw, the Sine is the function we will choose. This is because we have an equation,

$$\sin 20° = \frac{a}{45},$$

which can be solved for the missing variable a. If we used the Cosine function, we could find the length of the adjacent, but that would not get us the length of a.[xv] The Tangent function is even worse; we would have two variables in our equation, which would put us in an untenable position.

To solve this equation, we should evaluate the left side, and then get the variable all by itself. Using a calculator, we find that $\sin 20° \approx 0.342$. Hence

$$0.342 = \frac{a}{45}$$

$$15.39 \approx a.$$

The key to these problems is to identify the correct relationship. Then it's just a matter of substituting and solving, both of which are basic Algebra skills.

Example 4b

Determine the length of b given Figure 40.

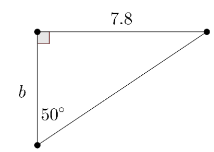


Figure 40

We face a similar problem. Here the best relationship to use is Tangent, since we can then form the equation

$$\tan 50° = \frac{7.8}{b}.$$

Then we just solve the equation whence

[xv] That being said, we could find the length of a after we find the length of the adjacent. It's not an ideal process though, as we have to go through the entire procedure again, or, alternatively, use Pythagoras' Theorem. Either way, it's much more efficient to use the Sine function.

$$b \approx 6.54.$$

Example 4c

Complete the right triangle shown in Figure 41.

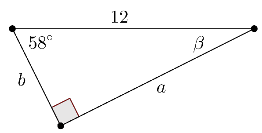


Figure 41

Recall that when we are asked to "complete" a right triangle, it just means that we should find out the length of every side and the measure of every angle.[xvi]

We have some freedom in how we approach this problem, but there are also some limitations.[xvii] Let us first find a. To do this, we must ask that ever-popular question: What is the relationship? In this case, we could[xviii] use the Sine function to get

$$\sin 58° = \frac{a}{12}$$
$$0.85 \approx \frac{a}{12}$$
$$a \approx 10.18.$$

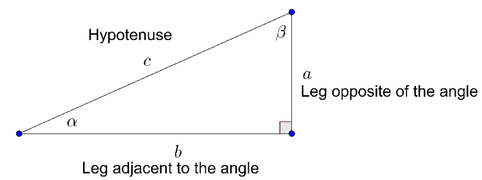
Now that we have a, we can easily find b by using Pythagoras' Theorem. We can also find β easily, since the sum of the three angles must be 180°. Hence

$$b \approx 6.35, \beta = 32°.$$

Before you begin practicing these concepts, there are three other Trig functions which exist. These are rarely used by themselves, but they will be of some importance later. We mention them here for the sake of completion.

[xvi] That makes a total of six things we're looking for, although we must be given at least three of them to determine the other three.
[xvii] For example, could we find any of the sides using Pythagoras' Theorem?
[xviii] We say "could" because we could also find the length of b by using the Cosine function. Could we use the Tangent function here? Why or why not?

The reciprocal Trigonometric functions



The Cosecant function: $\csc\alpha = \frac{c}{a}$

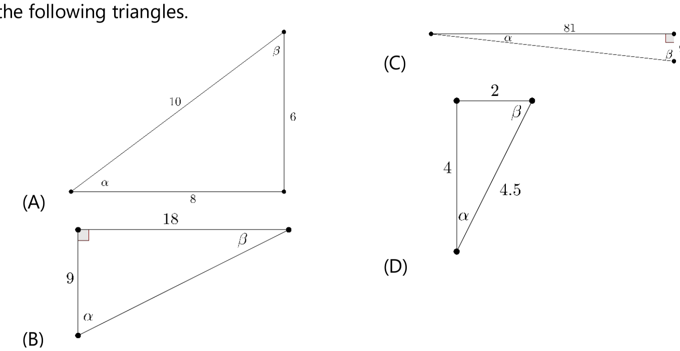
The Secant function: $\sec\alpha = \frac{c}{b}$

The Cotangent function: $\cot\alpha = \frac{b}{a}$

The name "reciprocal Trig functions" is very apropos. See if you can find out why.

§2 Exercises

1.) Write the ratios of the Sine, Cosine, and Tangent function (for both α and β) given the following triangles.



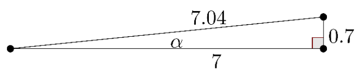
2.) Our next goal is to create a table for the Sine, Cosine, and Tangent functions. First, however, we'll only focus on those really nice angles which produce special right triangles.
   (A) What are the "really nice angles" of which we speak? (Hint: Think of the angles seen in all of our special right triangles.)

(B) Draw any two triangles with the information from (A). Make sure you label the sides and angles.

(C) Now create a table of values for the Sine, Cosine, and Tangent functions. Use (and expand) the incomplete table below as a model.

α	$\sin\alpha$
30°	$\frac{1}{2}$
45°	$\frac{\sqrt{2}}{2}$

3.) We have not yet defined $\sin 0°$ and $\cos 0°$, so let's do that now. Consider the triangle below, where α is a very small angle.



Note that, in the above case, $\sin\alpha$ will be relatively small (which you should verify).

(A) What will happen to the length of the side opposite of α as the angle of α decreases?

(B) Finally, imagine that $\alpha = 0$. What will the length of the side opposite to α be?

(C) Given your answer for (B), what must $\sin 0°$ be equal to?

(D) Now let's consider the Cosine function. Already, the length of the side adjacent to α and the length of the hypotenuse are very close. What is $\cos\alpha$ given the picture?

(E) As α gets close to zero, what will happen to the length of the hypotenuse in relation to the length of the adjacent side? (Hint: Try drawing smaller and smaller angles for α)

(F) Given your answer for (E), what must $\cos 0°$ be equal to?

(G) Verify your results for (C) and (F) with a calculator.

4.) Now let's define the results of $\sin 90°$ and $\cos 90°$.

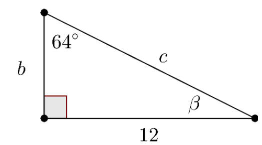
(A) We should try to use a similar approach as the previous problem. Draw a right triangle where α is close to 90°, but not quite that large.

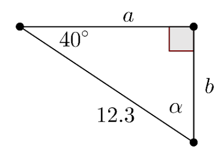
(B) Based on your picture, what will $\sin 90°$ and $\cos 90°$ be?

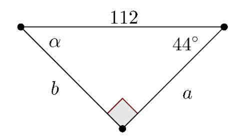
(C) Verify your results with a calculator.

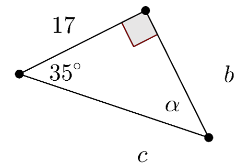
5.) Create a table of values for all three Trig functions from 0° to 90°. Increment each row by 5° (so your first number is 0°, then 5°, then 10°…). You should use a calculator for this problem.

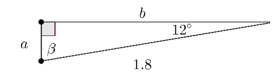
6.) Complete the following right triangles.

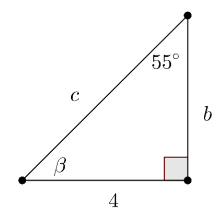
(A)


(B)


(C)


(D)


(E)


(F)


7.) The previous concepts can also be applied to quasi-real scenarios.

(A) Benny leans a ladder against a 24 foot wall, so that the top of the ladder touches the top of the wall perfectly. The angle that is created with the ground and the ladder is 70°. Determine the length of the ladder.

(B) Building A is shorter than Building B. To determine how much shorter, the mayor lays a tape measure from Building A to Building B and sees that the slant height is 45 m. Then he also figures out that the angle between the tape measure and Building B is 62°. How much shorter is Building A than Building B?

(C) A water slide is housed inside a tower that is 50 ft tall. The water slide travels in a straight line down toward the ground, where it hits the ground at a 24° angle. How much space is needed on the ground for this slide?

(D) A tree's shadow is 10 ft long. The angle that the sun creates with the flat ground is 80°. Determine the height of the tree.

8.) Create your own quasi-real situation, similar to one of the previous problems, that requires Trigonometry to solve.

9.) Look at the table of values you created for the Sine function.

(A) What is the maximum value of $\sin\alpha$?

(B) What is the minimum value of $\sin\alpha$?
(C) Are those maximum and minimum values true for any value of α? Try $\sin 500°$ or $\sin(-100°)$ to verify.
(D) Describe any patterns you see with the Sine function.

10.) Look at the table of values you created for the Cosine function.
(A) What is the maximum value of $\cos\alpha$?
(B) What is the minimum value of $\cos\alpha$?
(C) Are those maximum and minimum values true for any value of α?
(D) Describe any patterns you see with the Cosine function.

11.) Look at the table of values you created for the Tangent function.
(A) What is the maximum value of $\tan\alpha$?
(B) What is the minimum value of $\tan\alpha$?
(C) Are those maximum and minimum values true for any value of α?
(D) Describe any patterns you see with the Tangent function.

12.) Imagine for a moment that you measure a triangle in yards, and you find that the two legs measure 3 and 4 yards, while the hypotenuse measures 5 yards. You send this information to your friend who lives in a European country that normally uses meters instead.
(A) Calculate $\sin\alpha$ using the initial correct measurement that used yards.
(B) Now convert yards into meters (round to the nearest hundredth). Calculate $\sin\alpha$ in meters.
(C) Assume that when you sent this triangle to your friend, you did not label your measurements. So your friend assumes you mean 3 meters, 4 meters, and so on. Calculate $\sin\alpha$ with this incorrect information.
(D) Was your answer any different in (A), (B), or (C)? What does that tell you about Sine and Cosine ratios (viz., do units matter)?

13.) One popular mnemonic device for remembering the different Trig ratios is SOH CAH TOA. Explain what this means and how it helps you to remember how to set up your ratios. (If you don't know it, look it up)

14.) Let us now derive the Tangent function. To accomplish this, we only need the Sine and Cosine function.
(A) When deriving things, it is often useful to rewrite them in their most basic terms. Before we can do that, however, let us draw a right triangle. Choose any non-right angle and call it α. Then label the sides (according to your choice of α) as "opposite", "adjacent", and "hypotenuse".
(B) Using your picture, write out the Sine and Cosine ratios.
(C) Is there any way we can take our Sine and Cosine ratio and turn them into our Tangent ratio? Recall that Tangent is $\frac{\text{opposite}}{\text{adjacent}}$. Try adding, subtracting, multiplying, or dividing them.
(D) Why is it better to derive the tangent function than to define it? Put another way, what issues would you run into if you just defined everything in math?

§3 Relationships between the Trig functions

In our final section, we'll look at how the three main Trig functions are related to one another. We already know a few relationships, viz. that the Cosine function is the complement to the sine function. So it is fair to believe that there are other relationships that exist, and in this section, we will tease them out. We will spend our time deriving and formalizing many of the basic and fundamental **identities** in this section. An identity is a different way to write an equivalent statement. For example, $2 + 2 = 4$ is an identity; $2 + 2$ is simply a different name for 4.

Before we embark on learning some identities, let us first look at a more basic question: Does knowing one Trig ratio lead us to find any of the others? Put another way, if we know one Trig ratio, can we find *all* of the other?

Example 1a

If $\sin\alpha = \frac{3}{5}$, write the other six Trig ratios.

Let us first draw a picture. Because the Sine ratio is the side opposite to α to the hypotenuse, it makes sense to draw Figure 42.

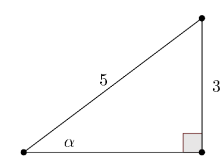


Figure 42

As before, this isn't the *only* triangle that you could draw.

This drawing makes our life much easier. Now what we'll do is determine the other ratios. Starting with Cosine, we would have

$$\cos\alpha = \frac{\text{adjacent}}{5}.$$

But what is the length of the adjacent side? This can be easily figured out with Pythagoras' Theorem, although, in this case, we recognize that we have a Pythagorean Triple, and thus the adjacent side is 4. With this in mind, the rest of the Trig ratios are elementary. They are

$$\cos\alpha = \frac{4}{5}, \qquad \tan\alpha = \frac{3}{4}, \qquad \csc\alpha = \frac{5}{3}, \qquad \sec\alpha = \frac{5}{4}, \qquad \cot\alpha = \frac{4}{3}.$$

Example 1b

If $\tan\beta = \frac{5}{7}$, write the other six Trig ratios.

You should, again, start with a picture.[i] We draw the triangle shown in Figure 43.

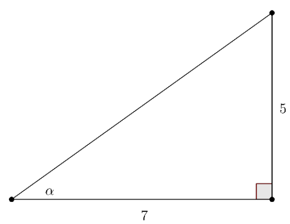


Figure 43

Before we write out any of our ratios, how about we find the length of the missing side, which in the present case is the hypotenuse. Since we do not recognize this as a Pythagorean Triple, we go ahead and use Pythagoras' Theorem. We find that

$$5^2 + 7^2 = c^2$$

$$c = \sqrt{74}.$$ [ii]

With this value, we can now write all six ratios:

$$\sin\alpha = \frac{5}{\sqrt{74}}, \qquad \cos\alpha = \frac{7}{\sqrt{74}}, \qquad \tan\alpha = \frac{5}{7}, \qquad \csc\alpha = \frac{\sqrt{74}}{5}, \qquad \sec\alpha = \frac{\sqrt{74}}{7}, \qquad \cot\alpha = \frac{7}{5}.$$

[i] Sensing a pattern yet?
[ii] You should always leave this number in exact form. Simplify it if you can.

Note that it isn't necessary to rationalize the denominators, although it might be good practice for you to do so. Also note that some textbooks and standardized tests will require you to do this, so make sure you know how to do this.

Example 1c

Given that $\sin\alpha = a$, find all six Trig ratios.

This seems more difficult than the previous problems. Do not let first appearances intimidate you – you just need to draw a picture. The only issue with our picture is that we only seem to have the length of one side, a. But recall that $a = \frac{a}{1}$, so we can draw our picture as seen in Figure 44.

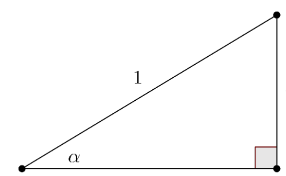


Figure 44

So what's the length of the adjacent side? We can figure this out using Pythagoras' Theorem, which, unsurprisingly, is exactly what we did in the previous Examples. We let b be the length of the adjacent side, and set up our equation like so:

$$a^2 + b^2 = 1^2.$$

We will let you finish this Example in the Exercises.

We have seen how we can relate one Trig ratio to the others. We will next relate the functions themselves.

A good starting point is to begin with what we know, and build from there. We know that

(1) $$\cos\alpha = \sin\beta \text{ iff } \alpha + \beta = 90°.$$

This is a fine relationship, but let's make this more useful. One issue is that there are two variables, so let's try to simplify that statement.

Example 2a

Rewrite (1) in terms of α.[iii]

We are given that $\cos\alpha = \sin\beta$ iff $\alpha + \beta = 90°$. We can thus make a substitution. Since

$$\alpha + \beta = 90°$$

then we know that

$$\beta = 90° - \alpha$$

and hence

$$\cos\alpha = \sin(90° - \alpha).$$

Example 2b

Given $\cos 20°$, what is the equivalent cofunction?

To answer this, we need only use the previous result. In the present case, $\alpha = 20°$ and hence

$$\cos 20° = \sin(90° - 20°) = \sin 70°.$$

The answer we're looking for is $\sin 70°$. A calculator can quickly verify that indeed, $\cos 20° = \sin 70°$.

Do recognize that the cofunction of Cosine is Sine, and vice versa. Which Trig function is the cofunction of Tangent? Or how about Cosecant? Is there an easy way to tell?

Example 2c

Write all of the **cofunctions identities**.

We will not write all of them, but instead, will write one of them and leave the rest to the reader.

$$\sin\alpha = \cos(90° - \alpha).$$

Other cofunction identities exist, and you will have to write them out in your exercises.

The cofunction identities can be useful, but mostly they just highlight the relationship between the Sine and Cosine function (and the other co-Trig functions).

[iii] In other words, there should only be one variable, α.

Let us now explore what happens when we square the Sine or Cosine functions. First of all, if we write $\sin\alpha^2$ there could be a bit of confusion.[iv] So, if we mean $(\sin\alpha)(\sin\alpha)$, we will write

$$\sin^2\alpha.$$

If we mean $\sin(\alpha^2)$, where the angle α is squared but not the function, we'll write it as $\sin(\alpha^2)$.

Example 3

Evaluate $\sin^2\alpha$ given Figure 45.

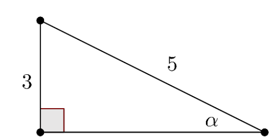


Figure 45

Recall that $\sin^2\alpha = (\sin\alpha)(\sin\alpha)$, and since $\sin\alpha = \frac{3}{5}$, we have, by substitution,

$$\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)$$

$$\frac{9}{25}.$$

You'll work to explore squared Trig functions in the exercises.

Now let's see what happens when we add two squared functions.

Let's try $\sin^2 45° + \cos^2 45°$. Since $\sin 45° = \frac{\sqrt{2}}{2}$, we can conclude that

$$\sin^2 45° = \frac{2}{4} = \frac{1}{2}.$$

This is also true of $\cos^2 45°$, since $\cos 45° = \frac{\sqrt{2}}{2}$. Thus we have

$$\sin^2 45° + \cos^2 45° = \frac{1}{2} + \frac{1}{2} = 1.$$

This is quite nice. Will this always be true?

[iv] Are we squaring the angle α or squaring the function $\sin\alpha$?

Example 4

Evaluate $\sin^2 60° + \cos^2 60°$.

From the previous section, we have memorized that $\sin 60° = \frac{\sqrt{3}}{2}$.[v] Thus $\sin^2 60° = \frac{3}{4}$. We also have memorized $\cos 60° = \frac{1}{2}$, and thus $\cos^2 60° = \frac{1}{4}$. We thusly conclude that

$$\sin^2 60° + \cos^2 60° = \frac{3}{4} + \frac{1}{4} = 1.$$

But that's strange… Why have we gotten the same answer? And why is that answer so pleasant? Have we stumbled upon some remarkable truth?

Pythagorean identity

For any angle α,

$$\sin^2 \alpha + \cos^2 \alpha = 1.$$

We'll revisit this identity in graphical format, but that will be after we reveal the Unit Circle. It is there that we will prove this identity, as it is very easy to do when we view it graphically.

It would be safe to ask if this is the only such relationship between cofunctions. For example, is $\tan^2 \alpha + \cot^2 \alpha$ also equal to one? You will explore that in the exercises.

In §2, we briefly introduced the Reciprocal Trig functions, such as Secant. These are defined as the reciprocals of their respective Trig function. In case, for example, we want to find the ratio of the hypotenuse to the opposite, we would then use the Cosecant function. And since

$$\csc \alpha = \frac{\text{hypotenuse}}{\text{opposite}}$$

while

$$\sin \alpha = \frac{\text{opposite}}{\text{hypotenuse}},$$

we recognize that these are reciprocals (thus the name).

Recall that one way we can write a reciprocal is to put it under one. Let's put that into symbols. If a is some number, and we want to find its reciprocal, we can just evaluate (or leave, if we prefer) $\frac{1}{a}$. So if we want to find the reciprocal of the number $\frac{3}{4}$, we just need to evaluate $\frac{1}{\frac{3}{4}}$, which is $\frac{4}{3}$, as you should verify.

[v] And if you haven't memorized it, that's ok too. You will then need to construct a $30° - 60° - 90°$ triangle and write out the ratio, then simplify what you have.

With this in mind we can now define the Reciprocal identities.

Reciprocal identities

$$\sin \alpha = \frac{1}{\csc \alpha}, \qquad \csc \alpha = \frac{1}{\sin \alpha}$$

Of course, there are other reciprocal identities, but you will write the remainder in your exercises.

Let us now put these identities to use.

Example 5a

Simplify $\frac{\sin \alpha}{\csc \alpha} + \frac{\cos \alpha}{\sec \alpha}$.

The key to this problem is to rewrite this expression into something easier. So what we'll want to do is identify some aspect of the expression that can be rewritten. In this case, let's try rewriting $\csc \alpha$ and see what we get. Since

$$\frac{\sin \alpha}{\csc \alpha} = \sin \alpha \cdot \frac{1}{\csc \alpha}$$

and $\frac{1}{\csc \alpha} = \sin \alpha$, we can say that

$$\frac{\sin \alpha}{\csc \alpha} = \sin^2 \alpha.$$

This same procedure will reveal that

$$\frac{\cos \alpha}{\sec \alpha} = \cos^2 \alpha.$$

So we can rewrite each addend in our original problem and we end up with

$$\sin^2 \alpha + \cos^2 \alpha,$$

which isn't too bad. But we can rewrite that expression as something even more simple! Since that is a Pythagorean identity and is equal to one, our final result is simply

$$1.$$

How did we know to change $\csc \alpha$ and $\sec \alpha$? Nothing but intuition – in other words, when we work with these sorts of problems, there is no prescribed method to simplify the expression. Experience is a big help, but do not underestimate planning and patience, either. And of course, you must be well-versed in the various identities that we've learned so far. You'll find that if you don't know the identities very well that simplifying expressions in this manner will be very difficult. No matter how good you become at these identities, however, be prepared to spend some time with them. Even very good mathematicians sometimes struggle with these, so don't feel incompetent if you don't get the answer right away.

One rule of thumb which we'll reiterate to you: You will most likely want to convert any Trig functions into Sine or Cosine, if possible. We have many identities which work with Sine and Cosine, but only a few that work with the reciprocal Trig functions. This isn't always the case, but it's usually the best place to start.

Example 5b

Simplify $\frac{\sin\alpha}{\cos\alpha}$.

You should have already found this out in the Exercises in the previous section, but this one is so important that going over it a second time will be helpful to you. Note that the procedure we use here will very rarely be used by you in your Exercises. But again, the result is so important we feel it necessary to include.

One way we can rewrite $\sin\alpha$ and $\cos\alpha$ is in terms of their ratios. But ratios require a triangle, right? And we don't have one, so what shall we do? Well, how about we make one? Consider Figure 46, which will allow us to find the ratios of our two Trig functions.

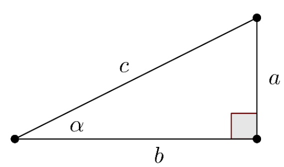


Figure 46

This isn't the only way you can draw this triangle, either. The key is just to draw one.

With this in mind, we can now substitute, since $\sin\alpha = \frac{a}{c}$ and $\cos\alpha = \frac{b}{c}$, we have

$$\frac{\sin\alpha}{\cos\alpha} = \frac{\frac{a}{c}}{\frac{b}{c}}.$$

Recall that one can divide two fractions by multiplying by the reciprocal. Thus we have

$$\frac{a}{c} \cdot \frac{c}{b}.$$

But $\frac{c}{c}$ is one, and therefore the c values cancel out. This leaves us with

$$\frac{a}{b}.$$

This is a ratio! And in fact, it is no more than the Tangent function's ratio. Thus we conclude that

$$\frac{\sin\alpha}{\cos\alpha} = \tan\alpha.$$

You will use this identity many, many times, and so we formalize it below.

Quotient identity

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$$

Example 5c

Simplify $\cos\alpha \cdot \tan\alpha$.

Following our rule of thumb, let's convert everything into Sines and Cosines. To do this, we simply rename $\tan\alpha$ using our identity above. We then have

$$\cos\alpha \cdot \frac{\sin\alpha}{\cos\alpha}$$

whence we see that the Cosines cancel. This simply leaves us with

$$\sin\alpha.$$

This process of simplifying expressions will be formalized in Unit six. Until then, the goal is to introduce you to this kind of thinking.

§3 Exercises

1.) Use the given Trig ratio to write out all six Trig ratios in each problem.

(A) $\sin\alpha = \frac{12}{13}$
(B) $\tan\alpha = \frac{4}{3}$
(C) $\cos\alpha = \frac{7}{25}$
(D) $\sin\alpha = \frac{1}{4}$
(E) $\cos\alpha = \frac{10}{13}$
(F) $\tan\alpha = \frac{5}{6}$
(G) $\csc\alpha = \frac{7}{2}$
(H) $\sec\alpha = \frac{10}{7}$

2.) Let us complete Example 1c. If $\sin \alpha = a$, what are the six Trig ratios?

3.) Now let $\tan \alpha = d$.
   (A) What are the six Trig ratios given this?
   (B) Compare this answer to the previous.

4.) In Example 1c, we let the length of the hypotenuse of the triangle to equal 1. Is that OK? Why don't we let the hypotenuse equal c, to allow for any and all possibilities? Now suppose $\sin \alpha = \frac{x}{c}$, and the lengths of your triangle are a, b, and c, with the hypotenuse equaling c.
   (A) What are the six Trig ratios?
   (B) Compare this with the previous two results.

5.) Given the Trig functions and angle measure, write the equivalent cofunction.
   (A) $\sin 30°$ (F) $\tan 14°$
   (B) $\cos 10°$ (G) $\csc 47°$
   (C) $\cot 7°$ (H) $\sin 25°$
   (D) $\sec 64°$ (I) $\tan(\beta + \gamma)$
   (E) $\cos 31°$ (J) $\sin \beta$

6.) Write out all of the cofunction identities, including the ones we discovered in the reading. Hint: There are six of them.

7.) Now write out all of the reciprocal identities. Hint: There are six of them.

8.) Evaluate the following.
   (A) $\sin^2 30°$ (F) $\sin^2 60°$
   (B) $\cos^2 30°$ (G) $\cos^2 60°$
   (C) $\tan^2 30°$ (H) $\tan^2 60°$
   (D) $\cos^2 45°$ (I) $\sin(30°^2)$
   (E) $\tan^2 45°$ (J) $\cos^2 \alpha$

9.) Write out a table of values for $\sin^2 \alpha$, $\cos^2 \alpha$, and $\tan^2 \alpha$, starting at $\alpha = 0$, going up by 5° each row, and ending at $\alpha = 90°$. You will need a calculator for this Exercise.

10.) Use your results from the previous Exercise to answer the following questions.
   (A) What is the maximum value of $\sin^2 \alpha$, $\cos^2 \alpha$, and $\tan^2 \alpha$?
   (B) What is the minimum value of $\sin^2 \alpha$, $\cos^2 \alpha$, and $\tan^2 \alpha$?
   (C) Are there any similarities or differences between $\sin^2 \alpha$ and $\sin \alpha$? Compare your results from the previous section.

11.) One of the most important relationships in Trigonometry is the Pythagorean Identity we discussed in the reading. Write this identity down now.

12.) Evaluate the following.
   (A) $\sin^2 60° + \cos^2 60°$ (C) $\sin^2\left(\frac{\alpha}{2}\right) + \cos^2\left(\frac{\alpha}{2}\right)$
   (B) $\cos^2 30° + \sin^2 30°$ (D) $\sin^2(3\alpha + \pi) + \cos^2(3\alpha + \pi)$

13.) It is often helpful to rewrite $\sin^2 \alpha$ or $\cos^2 \alpha$. Use the Pythagorean Identity to rewrite $\sin^2 \alpha$ and $\cos^2 \alpha$.

14.) Simplify the following.
   (A) $\tan \alpha \cdot \csc \alpha$
   (B) $(\sin \alpha + \cos \alpha)^2$

15.) Are there any other Pythagorean Identities? To find this out, use a calculator and try the following for different values of α.
   (A) $\sec^2 \alpha + \csc^2 \alpha$
   (B) $\tan^2 \alpha + \cot^2 \alpha$
   (C) There are two other Pythagorean Identities. First, using the previous two, guess what they might be. Then, if you can't figure it out, look them up and write them down now. We'll discover how to arrive at these results when we have some better tools.

16.) Answer True or False.
   (A) $\sin \alpha = \cos(\alpha - 90°)$
   (B) $\sin^2 \alpha + \cos^2 \beta = 1$ iff $\alpha + \beta = 90°$
   (C) $\sin^2 \alpha = \sin \alpha \cdot \alpha$
   (D) $\sin^2 \alpha$ is sometimes negative.[vi]

[vi] Assume $\alpha \in \mathbb{R}$.