11th Grade Lesson Plan Packet

4/27/2020-5/1/2020

GreatHearts Irving

Remote Learning Packet

NB: Please keep all work produced this week. Details regarding how to turn in this work will be forthcoming.

April 27-May 1, 2020
Course: 11 Art
Teacher(s): Ms. Frank clare.frank@greatheartsirving.org

Weekly Plan:

Monday, April 27
☐ Continue working on your project, with attention to craftsmanship in shading, mark-making and line quality.

Tuesday, April 28
☐ Sketchbook entry: Area of strength and area for improvement
☐ Continue working on your project

Wednesday, April 29
☐ Sketchbook entry: Distinctive qualities of your project
☐ Finish working on project

Thursday, April 30
☐ Photography Project: Converging Lines and Sense of Place

Friday, May 1
☐ Complete the formatting of your photography project. If you have time remaining, draw an image that represents your view onto the outside world, whether through a window, from a porch or balcony, or some other vantage point. Use any media you wish. Load a file of your image to the day's entry.

Devote 20 minutes of quality work time to your art assignments each day.

- *The assignments will be submitted as a single PDF of photos at the end of the week, with the exception of the photography project, which will be submitted as a Google Slides document.*
- *For written sketchbook entries you have two options:*
    a. *To write them out in your sketchbook as we do in class normally, and include them in the pdf upload at the end of the week*
    b. *To type them into a Google Doc assignment. This will be posted as an "ungraded" assignment, but really it's graded as part of the larger packet grade.*

Monday, April 27

1. Continue working on your project, with attention to craftsmanship in shading, mark-making and line quality. Keep in mind your individual expressive intentions as you work.

As you draw, remember to keep a clean folded piece of paper under your hand to avoid smudging.

Tuesday, April 28

1. In a dated sketchbook entry, address an area of strength and an area for improvement in your project.
2. Continue working on your project, using your self-evaluation to help direct your priorities as you work.

Wednesday, April 29

1. In a dated sketchbook entry, describe distinctive qualities of your project. What about the drawing style or composition speaks in a particular way of you, your touch, your interests, or your aesthetic judgment? There is a plethora of possibilities, and answering this question involves you looking thoughtfully and receptively at your own work.
2. Complete your project, looking at the image as a whole to achieve unity and harmony, to enhance visual interest, and to accentuate the individual qualities of your work with finishing touches.

Thursday, April 30

1. Review the project guidelines for the photographic project, Converging Lines and Sense of Place, and view the demo example on Google Classroom. (Instead of displaying your photographs in a handmade book, you will display them in a Google Slides slideshow presentation.)
2. Proceed to photograph your chosen environment, taking at least 10 quality photographs, from which you will select at least 6 (per guidelines).

Friday, May 1

1. Create a Google Slide document for "Converging Lines and Sense of Place" (ready for you on the assignment listing in Google Classroom). Create a title page and a body of 6-8 slides, followed by an end page, into which you could type a message, poem, short story or reflection relating to your project. Save into your GoogleDrive, and upload to the assignment on Google Classroom, then submit.

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Remote Learning Packet

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April 27 - May 1, 2020
Course: 11 Calculus I
Teacher(s): Mr. Simmons

Weekly Plan:

Monday, April 27
- [ ] Read my announcement on Google Classroom's stream
- [ ] Story time!
- [ ] Read about suprema
- [ ] Complete Problem 1
- [ ] Read the definition of an intersection

Tuesday, April 28
- [ ] Read Ch. 11 of Spivak until p. 179

Wednesday, April 29
- [ ] Read Ch. 11, pp. 179-186

Thursday, April 30
- [ ] Read Ch. 11, pp. 186-189
- [ ] Optional proof (ungraded)

Friday, May 1
- [ ] Complete Problem 11.6 ("Problem 6")

Monday, April 27

1. If you have not done so already, please read the long announcement that I posted on the stream (at 8am today).
2. Story time! If it's technologically feasible to do, please email me at least one sentence letting me know how you're doing, and tell me a fun story. I miss you (yes, you).

As a way of reviewing and solidifying some of the material from the past few weeks, we are going to be reading Spivak. Some of this is review, so if anyone is bored, I challenge you to prove each of Spivak's theorems yourself (before looking at Spivak's proof), and then completing as many as you can of the very fascinating problems he has at the end of each chapter. You'll enjoy them.

In reading Spivak, we are coming in at Ch. 11. He will employ two terms that we have not covered, but which will not take you longer than today's 40 minutes to become familiar with. In service of preparing us to read Chapter 11, spend today following these instructions:

3. Read about suprema (and infima) on pp. 116-18, starting at the beginning of the chapter and stopping at the statement of Theorem 7-1.
4. On a separate sheet of paper, complete Problem 8.1 (he calls it Problem 1, but to clarify which chapter it's in, I'm calling it Problem 8.1) at the end of Ch. 8. (You will use this same piece of paper the whole week, turning in just this one item next Sunday.)
5. Read the very brief definition of "intersection" in the middle of p. 43 (in the paragraph that starts, "If f and g are any two functions…").

Remember to read for understanding. That means possibly pausing on a single sentence and thinking about it for a while, maybe drawing a few diagrams to help you understand it. Don't rush yourself, or you'll simply struggle more later.

Tuesday, April 28

1. Read Ch. 11, from the beginning of the chapter all the way through the proof of Rolle's Theorem on p. 179. Read for understanding. (At times, Spivak will write something like, "I leave the remainder of this proof as an exercise for the reader." Don't feel obliged to complete those problems (but of course feel free to!).)

Though Spivak doesn't label it as such, his Theorem 11.2 (he calls it Theorem 2, but to clarify which chapter it's in, I'm calling it Theorem 11.2) is Fermat's Theorem.

I am aware that he has a slightly different definition of "critical point" from the one we learned. He only mentions points where f prime is zero, whereas we had previously included also points where f prime is undefined (so long as f itself was defined). We will be going with Spivak's definition, as we will with everything from now on.

Wednesday, April 29

1. Read Ch. 11, until Theorem 11.5 (“Theorem 5”) on p. 186. (At times, Spivak will write something like, “I leave the remainder of this proof as an exercise for the reader.” Don’t feel obliged to complete those problems (but of course feel free to!).)

Thursday, April 30

1. Read Ch. 11, through p. 189.
2. (Optional, ungraded) Page 189 ends with a statement of L’Hopital’s Rule (LOE-pee-Tall’s). If you would like to, try to prove it. I’m happy to read your proof and give you feedback. (If you’re going to prove it yourself, careful not to glance at spoilers on p. 190.)

Friday, May 1

1. Finish reading Ch. 11 (not including the appendix on convexity and concavity, unless you want to). (At times, Spivak will write something like, “I leave the remainder of this proof as an exercise for the reader.” Don’t feel obliged to complete those problems (but of course feel free to!).)
2. Answer Problem 11.6 (“Problem 6”), writing on Monday’s sheet of paper.
3. If you have extra time of the 40 minutes allotted for math today, spend it mastering the vocabulary from Ch. 11

CHAPTER 8 LEAST UPPER BOUNDS

This chapter reveals the most important property of the real numbers. Nevertheless, it is merely a sequel to Chapter 7; the path which must be followed has already been indicated, and further discussion would be useless delay.

DEFINITION

A set A of real numbers is **bounded above** if there is a number x such that

$$x \geq a \quad \text{for every } a \text{ in } A.$$

Such a number x is called an **upper bound** for A.

Obviously A is bounded above if and only if there is a number x which is an upper bound for A (and in this case there will be lots of upper bounds for A); we often say, as a concession to idiomatic English, that "A has an upper bound" when we mean that there is a number which is an upper bound for A.

Notice that the term "bounded above" has now been used in two ways—first, in Chapter 7, in reference to functions, and now in reference to sets. This dual usage should cause no confusion, since it will always be clear whether we are talking about a set of numbers or a function. Moreover, the two definitions are closely connected: if A is the set $\{f(x)\colon a \leq x \leq b\}$, then the function f is bounded above on $[a, b]$ if and only if the set A is bounded above.

The entire collection $\mathbf{R}$ of real numbers, and the natural numbers $\mathbf{N}$, are both examples of sets which are *not* bounded above. An example of a set which *is* bounded above is

$$A = \{x\colon 0 \leq x < 1\}.$$

To show that A is bounded above we need only name some upper bound for A, which is easy enough; for example, 138 is an upper bound for A, and so are 2, $1\frac{1}{2}$, $1\frac{1}{4}$, and 1. Clearly, 1 is the least upper bound of A; although the phrase just introduced is self-explanatory, in order to avoid any possible confusion (in particular, to ensure that we all know what the superlative of "less" means), we define this explicitly.

DEFINITION

A number x is a **least upper bound** of A if

(1) x is an upper bound of A,
and (2) if y is an upper bound of A, then $x \leq y$.

The use of the indefinite article "a" in this definition was merely a concession to temporary ignorance. Now that we have made a precise definition, it is easily seen that if x and y are both least upper bounds of A, then $x = y$. Indeed, in this case

$x \leq y$, since y is an upper bound, and x is a least upper bound,
and $y \leq x$, since x is an upper bound, and y is a least upper bound;

it follows that $x = y$. For this reason we speak of *the* least upper bound of A. The term **supremum** of A is synonymous and has one advantage. It abbreviates quite nicely to

$$\sup A \qquad \text{(pronounced "soup A")}$$

and saves us from the abbreviation

$$\text{lub } A$$

(which is nevertheless used by some authors).

There is a series of important definitions, analogous to those just given, which can now be treated more briefly. A set A of real numbers is **bounded below** if there is a number x such that

$$x \leq a \quad \text{for every } a \text{ in } A.$$

Such a number x is called a **lower bound** for A. A number x is the **greatest lower bound** of A if

(1) x is a lower bound of A,
and (2) if y is a lower bound of A, then $x \geq y$.

The greatest lower bound of A is also called the **infimum** of A, abbreviated

$$\inf A;$$

some authors use the abbreviation

$$\text{glb } A.$$

One detail has been omitted from our discussion so far—the question of which sets have at least one, and hence exactly one, least upper bound or greatest lower bound. We will consider only least upper bounds, since the question for greatest lower bounds can then be answered easily (Problem 2).

If A is not bounded above, then A has no upper bound at all, so A certainly cannot be expected to have a least upper bound. It is tempting to say that A does have a least upper bound if it has *some* upper bound, but, like the principle of mathematical induction, this assertion can fail to be true in a rather special way. If $A = \emptyset$, then A is bounded above. Indeed, any number x is an upper bound for $\emptyset$:

$$x \geq y \quad \text{for every } y \text{ in } \emptyset,$$

simply because there is no y in $\emptyset$. Since *every* number is an upper bound for $\emptyset$, there is surely no least upper bound for $\emptyset$. With this trivial exception however,

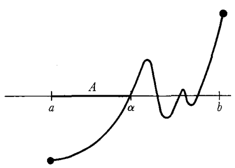


FIGURE 1

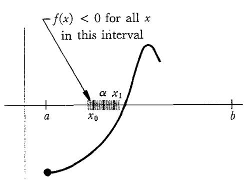


FIGURE 2

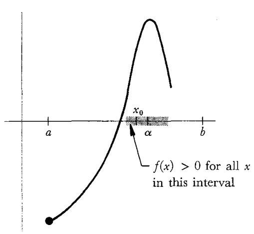


FIGURE 3

our assertion is true—and very important, definitely important enough to warrant consideration of details. We are finally ready to state the last property of the real numbers which we need.

(P13) (The least upper bound property) If A is a set of real numbers, $A \neq \emptyset$, and A is bounded above, then A has a least upper bound.

Property P13 may strike you as anticlimactic, but that is actually one of its virtues. To complete our list of basic properties for the real numbers we require no particularly abstruse proposition, but only a property so simple that we might feel foolish for having overlooked it. Of course, the least upper bound property is not really so innocent as all that; after all, it does *not* hold for the rational numbers $\mathbf{Q}$. For example, if A is the set of all rational numbers x satisfying $x^2 < 2$, then there is no *rational* number y which is an upper bound for A and which is less than or equal to every other *rational* number which is an upper bound for A. It will become clear only gradually how significant P13 is, but we are already in a position to demonstrate its power, by supplying the proofs which were omitted in Chapter 7.

THEOREM 7-1 If f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$, then there is some number x in $[a, b]$ such that $f(x) = 0$.

PROOF Our proof is merely a rigorous version of the outline developed at the end of Chapter 7—we will locate the smallest number x in $[a, b]$ with $f(x) = 0$.

Define the set A, shown in Figure 1, as follows:

$$A = \{x: a \leq x \leq b, \text{ and } f \text{ is negative on the interval } [a, x]\}.$$

Clearly $A \neq \emptyset$, since a is in A; in fact, there is some $\delta > 0$ such that A contains all points x satisfying $a \leq x < a + \delta$; this follows from Problem 6-15, since f is continuous on $[a, b]$ and $f(a) < 0$. Similarly, b is an upper bound for A and, in fact, there is a $\delta > 0$ such that all points x satisfying $b - \delta < x \leq b$ are upper bounds for A; this also follows from Problem 6-15, since $f(b) > 0$.

From these remarks it follows that A has a least upper bound α and that $a < \alpha < b$. We now wish to show that $f(\alpha) = 0$, by eliminating the possibilities $f(\alpha) < 0$ and $f(\alpha) > 0$.

Suppose first that $f(\alpha) < 0$. By Theorem 6-3, there is a $\delta > 0$ such that $f(x) < 0$ for $\alpha - \delta < x < \alpha + \delta$ (Figure 2). Now there is some number x_0 in A which satisfies $\alpha - \delta < x_0 < \alpha$ (because otherwise α would not be the *least* upper bound of A). This means that f is negative on the whole interval $[a, x_0]$. But if x_1 is a number between α and $\alpha + \delta$, then f is also negative on the whole interval $[x_0, x_1]$. Therefore f is negative on the interval $[a, x_1]$, so x_1 is in A. But this contradicts the fact that a is an upper bound for A; our original assumption that $f(\alpha) < 0$ must be false.

Suppose, on the other hand, that $f(\alpha) > 0$. Then there is a number $\delta > 0$ such that $f(x) > 0$ for $\alpha - \delta < x < \alpha + \delta$ (Figure 3). Once again we know that there is an x_0 in A satisfying $\alpha - \delta < x_0 < \alpha$; but this means that f is negative on $[a, x_0]$, which is impossible, since $f(x_0) > 0$. Thus the assumption

$$(12) \quad f(x) = \sin^2(\sin(\sin^2(x \sin^2 x^2))) \cdot \sin\left(\frac{x + \sin(x \sin x)}{x + \sin x}\right).$$

By what criterion, you may feel impelled to ask, can such functions, especially a monstrosity like (12), be considered simple? The answer is that they can be built up from a few simple functions using a few simple means of combining functions. In order to construct the functions (9)–(12) we need to start with the "identity function" I, for which $I(x) = x$, and the "sine function" sin, whose value $\sin(x)$ at x is often written simply $\sin x$. The following are some of the important ways in which functions may be combined to produce new functions.

If f and g are any two functions, we can define a new function $f + g$, called the **sum** of f and g, by the equation

$$(f + g)(x) = f(x) + g(x).$$

Note that according to the conventions we have adopted, the domain of $f + g$ consists of all x for which "$f(x) + g(x)$" makes sense, i.e., the set of all x in both domain f and domain g. If A and B are any two sets, then $A \cap B$ (read "A intersect B" or "the intersection of A and B") denotes the set of x in both A and B; this notation allows us to write domain $(f + g) =$ domain $f \cap$ domain g.

In a similar vein, we define the **product** $f \cdot g$ and the **quotient** $\frac{f}{g}$ (or f/g) of f and g by

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

and

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$

Moreover, if g is a function and c is a number, we define a new function $c \cdot g$ by

$$(c \cdot g)(x) = c \cdot g(x).$$

This becomes a special case of the notation $f \cdot g$ if we agree that the symbol c should also represent the function f defined by $f(x) = c$; such a function, which has the same value for all numbers x, is called a **constant function.**

The domain of $f \cdot g$ is domain $f \cap$ domain g, and the domain of $c \cdot g$ is simply the domain of g. On the other hand, the domain of f/g is rather complicated—it may be written domain $f \cap$ domain $g \cap \{x: g(x) \neq 0\}$, the symbol $\{x: g(x) \neq 0\}$ denoting the set of numbers x such that $g(x) \neq 0$. In general, $\{x: \ldots\}$ denotes the set of all x such that "$\ldots$" is true. Thus $\{x: x^3 + 3 < 11\}$ denotes the set of all numbers x such that $x^3 < 8$, and consequently $\{x: x^3 + 3 < 11\} = \{x: x < 2\}$. Either of these symbols could just as well have been written using y everywhere instead of x. Variations of this notation are common, but hardly require any discussion. Any one can guess that $\{x > 0: x^3 < 8\}$ denotes the set of positive numbers whose cube is less than 8; it could be expressed more formally as $\{x: x > 0 \text{ and } x^3 < 8\}$.

CHAPTER 11 SIGNIFICANCE OF THE DERIVATIVE

One aim in this chapter is to justify the time we have spent learning to find the derivative of a function. As we shall see, knowing just a little about f' tells us a lot about f. Extracting information about f from information about f' requires some difficult work, however, and we shall begin with the one theorem which is really easy.

This theorem is concerned with the maximum value of a function on an interval. Although we have used this term informally in Chapter 7, it is worthwhile to be precise, and also more general.

DEFINITION Let f be a function and A a set of numbers contained in the domain of f. A point x in A is a **maximum point** for f on A, if

$$f(x) \geq f(y) \quad \text{for every } y \text{ in } A.$$

The number $f(x)$ itself is called the **maximum value** of f on A (and we also say that f "has its maximum value on A at x").

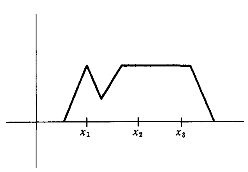


FIGURE 1

Notice that the maximum value of f on A could be $f(x)$ for several different x (Figure 1); in other words, a function f can have several different maximum points on A, although it can have at most one maximum value. Usually we shall be interested in the case where A is a closed interval $[a, b]$; if f is continuous, then Theorem 7-3 guarantees that f does indeed have a maximum value on $[a, b]$.

The definition of a minimum of f on A will be left to you. (One possible definition is the following: f has a minimum on A at x, if $-f$ has a maximum on A at x.)

We are now ready for a theorem which does not even depend upon the existence of least upper bounds.

THEOREM 1 Let f be any function defined on (a, b). If x is a maximum (or a minimum) point for f on (a, b), and f is differentiable at x, then $f'(x) = 0$.
(Notice that we do not assume differentiability, or even continuity, of f at other points.)

PROOF Consider the case where f has a maximum at x. (Figure 2 illustrates the simple idea behind the whole argument—secants drawn through points to the left of $(x, f(x))$ have slopes ≥ 0, and secants drawn through points to the right of $(x, f(x))$ have slopes ≤ 0.) Analytically, this argument proceeds as follows.

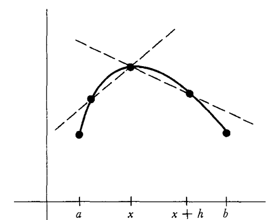


FIGURE 2

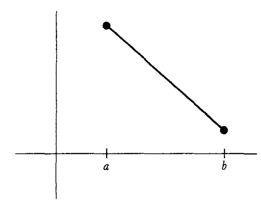


FIGURE 3

If h is any number such that $x + h$ is in (a, b), then

$$f(x) \geq f(x + h),$$

since f has a maximum on (a, b) at x. This means that

$$f(x + h) - f(x) \leq 0.$$

Thus, if $h > 0$ we have

$$\frac{f(x + h) - f(x)}{h} \leq 0,$$

and consequently

$$\lim_{h \to 0^+} \frac{f(x + h) - f(x)}{h} \leq 0.$$

On the other hand, if $h < 0$, we have

$$\frac{f(x + h) - f(x)}{h} \geq 0,$$

so

$$\lim_{h \to 0^-} \frac{f(x + h) - f(x)}{h} \geq 0.$$

By hypothesis, f is differentiable at x, so these two limits must be equal, in fact equal to $f'(x)$. This means that

$$f'(x) \leq 0 \quad \text{and} \quad f'(x) \geq 0,$$

from which it follows that $f'(x) = 0$.

The case where f has a minimum at x is left to you (give a one-line proof). ▌

Notice (Figure 3) that we cannot replace (a, b) by $[a, b]$ in the statement of the theorem (unless we add to the hypothesis the condition that x is in (a, b)).

Since $f'(x)$ depends only on the values of f near x, it is almost obvious how to get a stronger version of Theorem 1. We begin with a definition which is illustrated in Figure 4.

DEFINITION Let f be a function, and A a set of numbers contained in the domain of f. A point x in A is a **local maximum [minimum] point** for f on A if there is some $\delta > 0$ such that x is a maximum [minimum] point for f on $A \cap (x - \delta, x + \delta)$.

THEOREM 2 If f is defined on (a, b) and has a local maximum (or minimum) at x, and f is differentiable at x, then $f'(x) = 0$.

PROOF You should see why this is an easy application of Theorem 1. ▌

The converse of Theorem 2 is definitely not true—it is possible for $f'(x)$ to be 0 even if x is not a local maximum or minimum point for f. The simplest example is provided by the function $f(x) = x^3$; in this case $f'(0) = 0$, but f has no local maximum or minimum anywhere.

Probably the most widespread misconceptions about calculus are concerned with the behavior of a function f near x when $f'(x) = 0$. The point made in the previous paragraph is so quickly forgotten by those who want the world to be simpler than it is, that we will repeat it: the converse of Theorem 2 is *not* true—the condition $f'(x) = 0$ does *not* imply that x is a local maximum or minimum point of f. Precisely for this reason, special terminology has been adopted to describe numbers x which satisfy the condition $f'(x) = 0$.

DEFINITION

A **critical point** of a function f is a number x such that

$$f'(x) = 0.$$

The number $f(x)$ itself is called a **critical value** of f.

The critical values of f, together with a few other numbers, turn out to be the ones which must be considered in order to find the maximum and minimum of a given function f. To the uninitiated, finding the maximum and minimum value of a function represents one of the most intriguing aspects of calculus, and there is no denying that problems of this sort are fun (until you have done your first hundred or so).

Let us consider first the problem of finding the maximum or minimum of f on a closed interval $[a, b]$. (Then, if f is continuous, we can at least be sure that a maximum and minimum value exist.) In order to locate the maximum and minimum of f three kinds of points must be considered:

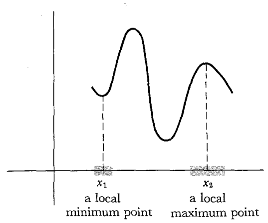


FIGURE 4

(1) The critical points of f in $[a, b]$.
(2) The end points a and b.
(3) Points x in $[a, b]$ such that f is not differentiable at x.

If x is a maximum point or a minimum point for f on $[a, b]$, then x must be in one of the three classes listed above: for if x is not in the second or third group, then x *is* in (a, b) and f *is* differentiable at x; consequently $f'(x) = 0$, by Theorem 1, and this means that x is in the first group.

If there are many points in these three categories, finding the maximum and minimum of f may still be a hopeless proposition, but when there are only a few critical points, and only a few points where f is not differentiable, the procedure is fairly straightforward: one simply finds $f(x)$ for each x satisfying $f'(x) = 0$, and $f(x)$ for each x such that f is not differentiable at x and, finally, $f(a)$ and $f(b)$. The biggest of these will be the maximum value of f, and the smallest will be the minimum. A simple example follows.

Suppose we wish to find the maximum and minimum value of the function

$$f(x) = x^3 - x$$

on the interval $[-1, 2]$. To begin with, we have

$$f'(x) = 3x^2 - 1,$$

so $f'(x) = 0$ when $3x^2 - 1 = 0$, that is, when

$$x = \sqrt{1/3} \quad \text{or} \quad -\sqrt{1/3}.$$

The numbers $\sqrt{1/3}$ and $-\sqrt{1/3}$ both lie in $[-1, 2]$, so the first group of candidates for the location of the maximum and the minimum is

$$(1) \quad \sqrt{1/3}, -\sqrt{1/3}.$$

The second group contains the end points of the interval

$$(2) \quad -1, 2.$$

The third group is empty, since f is differentiable everywhere. The final step is to compute

$$\begin{aligned} f(\sqrt{1/3}) &= (\sqrt{1/3})^3 - \sqrt{1/3} = \tfrac{1}{3}\sqrt{1/3} - \sqrt{1/3} = -\tfrac{2}{3}\sqrt{1/3}, \\ f(-\sqrt{1/3}) &= (-\sqrt{1/3})^3 - (-\sqrt{1/3}) = -\tfrac{1}{3}\sqrt{1/3} + \sqrt{1/3} = \tfrac{2}{3}\sqrt{1/3}, \\ f(-1) &= 0, \\ f(2) &= 6. \end{aligned}$$

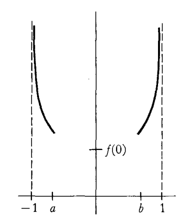


FIGURE 5

Clearly the minimum value is $-\tfrac{2}{3}\sqrt{1/3}$, occurring at $\sqrt{1/3}$, and the maximum value is 6, occurring at 2.

This sort of procedure, if feasible, will always locate the maximum and minimum value of a continuous function on a closed interval. If the function we are dealing with is not continuous, however, or if we are seeking the maximum or minimum on an open interval or the whole line, then we cannot even be sure beforehand that the maximum and minimum values exist, so all the information obtained by this procedure may say nothing. Nevertheless, a little ingenuity will often reveal the nature of things. In Chapter 7 we solved just such a problem when we showed that if n is even, then the function

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

has a minimum value on the whole line. This proves that the minimum value must occur at some number x satisfying

$$0 = f'(x) = nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_0.$$

If we can solve this equation, and compare the values of $f(x)$ for such x, we can actually find the minimum of f. One more example may be helpful. Suppose we wish to find the maximum and minimum, if they exist, of the function

$$f(x) = \frac{1}{1 - x^2}$$

on the open interval $(-1, 1)$. We have

$$f'(x) = \frac{2x}{(1 - x^2)^2},$$

so $f'(x) = 0$ only for $x = 0$. We can see immediately that for x close to 1 or -1 the values of $f(x)$ become arbitrarily large, so f certainly does not have a maximum. This observation also makes it easy to show that f has a minimum at 0. We just note (Figure 5) that there will be numbers a and b, with

$$-1 < a < 0 \quad \text{and} \quad 0 < b < 1,$$

such that $f(x) > f(0)$ for

$$-1 < x \leq a \quad \text{and} \quad b \leq x < 1.$$

This means that the minimum of f on $[a, b]$ is the minimum of f on all of $(-1, 1)$. Now on $[a, b]$ the minimum occurs either at 0 (the only place where $f' = 0$), or at a or b, and a and b have already been ruled out, so the minimum value is $f(0) = 1$.

In solving these problems we purposely did not draw the graphs of $f(x) = x^3 - x$ and $f(x) = 1/(1 - x^2)$, but it is not cheating to draw the graph (Figure 6) as long as you do not rely solely on your picture to prove anything. As a matter of fact, we are now going to discuss a method of sketching the graph of a function that really gives enough information to be used in discussing maxima and minima—in fact we will be able to locate even *local* maxima and minima. This method involves consideration of the sign of $f'(x)$, and relies on some deep theorems.

The theorems about derivatives which have been proved so far, always yield information about f' in terms of information about f. This is true even of Theorem 1, although this theorem can sometimes be used to determine certain information about f, namely, the location of maxima and minima. When the derivative was first introduced, we emphasized that $f'(x)$ is not $[f(x + h) - f(x)]/h$ for any particular h, but only a limit of these numbers as h approaches 0; this fact becomes painfully relevant when one tries to extract information about f from information about f'. The simplest and most frustrating illustration of the difficulties encountered is afforded by the following question: If $f'(x) = 0$ for all x, must f be a constant function? It is impossible to imagine how f could be anything else, and this conviction is strengthened by considering the physical interpretation—if the velocity of a particle is always 0, surely the particle must be standing still! Nevertheless it is difficult even to begin a proof that only the constant functions satisfy $f'(x) = 0$ for all x. The hypothesis $f'(x) = 0$ only means that

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = 0,$$

and it is not at all obvious how one can use the information about the limit to derive information about the function.

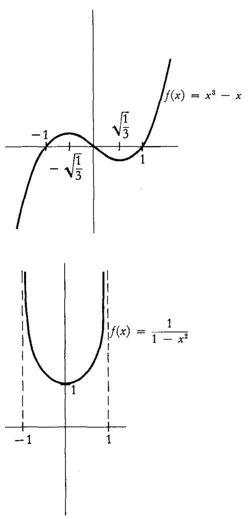


FIGURE 6

The fact that f is a constant function if $f'(x) = 0$ for all x, and many other facts of the same sort, can all be derived from a fundamental theorem, called the Mean Value Theorem, which states much stronger results. Figure 7 makes it plausible that if f is differentiable on $[a, b]$, then there is some x in (a, b) such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

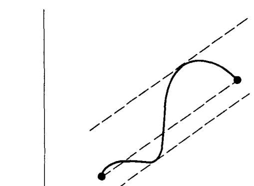


FIGURE 7

Geometrically this means that some tangent line is parallel to the line between $(a, f(a))$ and $(b, f(b))$. The Mean Value Theorem asserts that this is true—there is some x in (a, b) such that $f'(x)$, the instantaneous rate of change of f at x, is exactly equal to the average or "mean" change of f on $[a, b]$, this average change being $[f(b) - f(a)]/[b - a]$. (For example, if you travel 60 miles in one hour, then at some time you must have been traveling exactly 60 miles per hour.) This theorem is one of the most important theoretical tools of calculus—probably the deepest result about derivatives. From this statement you might conclude that the proof is difficult, but there you would be wrong—the hard theorems in this book have occurred long ago, in Chapter 7. It is true that if you try to prove the Mean Value Theorem yourself you will probably fail, but this is neither evidence that the theorem is hard, nor something to be ashamed of. The first proof of the theorem was an achievement, but today we can supply a proof which is quite simple. It helps to begin with a very special case.

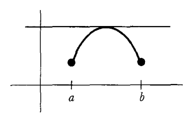


FIGURE 8

THEOREM 3 (ROLLE'S THEOREM) If f is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, then there is a number x in (a, b) such that $f'(x) = 0$.

PROOF It follows from the continuity of f on $[a, b]$ that f has a maximum and a minimum value on $[a, b]$.

Suppose first that the maximum value occurs at a point x in (a, b). Then $f'(x) = 0$ by Theorem 1, and we are done (Figure 8).

Suppose next that the minimum value of f occurs at some point x in (a, b). Then, again, $f'(x) = 0$ by Theorem 1 (Figure 9).

Finally, suppose the maximum and minimum values both occur at the end points. Since $f(a) = f(b)$, the maximum and minimum values of f are equal, so f is a constant function (Figure 10), and for a constant function we can choose any x in (a, b). ▮

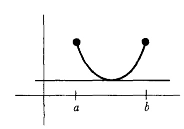


FIGURE 9

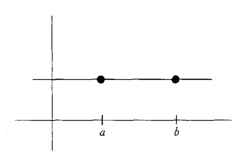


FIGURE 10

Notice that we really needed the hypothesis that f is differentiable everywhere on (a, b) in order to apply Theorem 1. Without this assumption the theorem is false (Figure 11).

You may wonder why a special name should be attached to a theorem as easily proved as Rolle's Theorem. The reason is, that although Rolle's Theorem is a special case of the Mean Value Theorem, it also yields a simple proof of the Mean Value Theorem. In order to prove the Mean Value

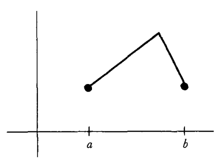


FIGURE 11

Theorem we will apply Rolle's Theorem to the function which gives the length of the vertical segment shown in Figure 12; this is the difference between $f(x)$, and the height at x of the line L between $(a, f(a))$ and $(b, f(b))$. Since L is the graph of

$$g(x) = \left[\frac{f(b) - f(a)}{b - a}\right](x - a) + f(a),$$

we want to look at

$$f(x) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a) - f(a).$$

As it turns out, the constant $f(a)$ is irrelevant.

THEOREM 4 (THE MEAN VALUE THEOREM) If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a number x in (a, b) such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

PROOF Let

$$h(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a).$$

Clearly, h is continuous on $[a, b]$ and differentiable on (a, b), and

$$\begin{aligned} h(a) &= f(a), \\ h(b) &= f(b) - \left[\frac{f(b) - f(a)}{b - a}\right](b - a) \\ &= f(a). \end{aligned}$$

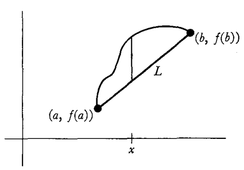


FIGURE 12

Consequently, we may apply Rolle's Theorem to h and conclude that there is some x in (a, b) such that

$$0 = h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

so that

$$f'(x) = \frac{f(b) - f(a)}{b - a}. \blacksquare$$

Notice that the Mean Value Theorem still fits into the pattern exhibited by previous theorems—information about f yields information about f'. This information is so strong, however, that we can now go in the other direction.

COROLLARY 1 If f is defined on an interval and $f'(x) = 0$ for all x in the interval, then f is constant on the interval.

PROOF Let a and b be any two points in the interval with $a \neq b$. Then there is some x

in (a, b) such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

But $f'(x) = 0$ for all x in the interval, so

$$0 = \frac{f(b) - f(a)}{b - a},$$

and consequently $f(a) = f(b)$. Thus the value of f at any two points in the interval is the same, i.e., f is constant on the interval. $\blacksquare$

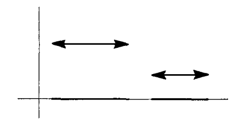


FIGURE 13

Naturally, Corollary 1 does not hold for functions defined on two or more intervals (Figure 13).

COROLLARY 2 If f and g are defined on the same interval, and $f'(x) = g'(x)$ for all x in the interval, then there is some number c such that $f = g + c$.

PROOF For all x in the interval we have $(f - g)'(x) = f'(x) - g'(x) = 0$ so, by Corollary 1, there is a number c such that $f - g = c$. $\blacksquare$

The statement of the next corollary requires some terminology, which is illustrated in Figure 14.

DEFINITION A function f is **increasing** on an interval if $f(a) < f(b)$ whenever a and b are two numbers in the interval with $a < b$. The function f is **decreasing** on an interval if $f(a) > f(b)$ for all a and b in the interval with $a < b$. (We often say simply that f is increasing or decreasing, in which case the interval is understood to be the domain of f.)

COROLLARY 3 If $f'(x) > 0$ for all x in an interval, then f is increasing on the interval; if $f'(x) < 0$ for all x in the interval, then f is decreasing on the interval.

PROOF Consider the case where $f'(x) > 0$. Let a and b be two points in the interval with $a < b$. Then there is some x in (a, b) with

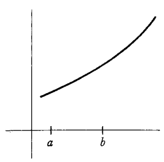
$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

But $f'(x) > 0$ for all x in (a, b), so

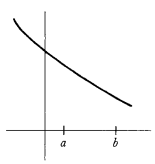
$$\frac{f(b) - f(a)}{b - a} > 0.$$

Since $b - a > 0$ it follows that $f(b) > f(a)$.

The proof when $f'(x) < 0$ for all x is left to you. $\blacksquare$



(a) an increasing function



(b) a decreasing function

FIGURE 14

Notice that although the converses of Corollary 1 and Corollary 2 are true (and obvious), the converse of Corollary 3 is not true. If f is increasing, it is easy to see that $f'(x) \cancel{>} \geq 0$ for all x, but the equality sign might hold for some x (consider $f(x) = x^3$).

Corollary 3 provides enough information to get a good idea of the graph of a function with a minimal amount of point plotting. Consider, once more, the function $f(x) = x^3 - x$. We have

$$f'(x) = 3x^2 - 1.$$

We have already noted that $f'(x) = 0$ for $x = \sqrt{1/3}$ and $x = -\sqrt{1/3}$, and it is also possible to determine the sign of $f'(x)$ for all other x. Note that $3x^2 - 1 > 0$ precisely when

$$3x^2 > 1,$$
$$x^2 > \tfrac{1}{3},$$
$$x > \sqrt{1/3} \quad \text{or} \quad x < -\sqrt{1/3};$$

thus $3x^2 - 1 < 0$ precisely when

$$-\sqrt{1/3} < x < \sqrt{1/3}.$$

Thus f is increasing for $x < -\sqrt{1/3}$, decreasing between $-\sqrt{1/3}$ and $\sqrt{1/3}$, and once again increasing for $x > \sqrt{1/3}$. Combining this information with the following facts

(1) $f(-\sqrt{1/3}) = \frac{2}{3}\sqrt{1/3}$,
$f(\sqrt{1/3}) = -\frac{2}{3}\sqrt{1/3}$,
(2) $f(x) = 0$ for $x = -1, 0, 1$,
(3) $f(x)$ gets large as x gets large, and large negative as x gets large negative,

it is possible to sketch a pretty respectable approximation to the graph (Figure 15).

By the way, notice that the intervals on which f increases and decreases could have been found without even bothering to examine the sign of f'. For example, since f' is continuous, and vanishes only at $-\sqrt{1/3}$ and $\sqrt{1/3}$, we know that f' always has the same sign on the interval $(-\sqrt{1/3}, \sqrt{1/3})$. Since $f(-\sqrt{1/3}) > f(\sqrt{1/3})$, it follows that f decreases on this interval. Similarly, f' always has the same sign on $(\sqrt{1/3}, \infty)$ and $f(x)$ is large for large x, so f must be increasing on $(\sqrt{1/3}, \infty)$. Another point worth noting: If f' is continuous, then the sign of f' on the interval between two adjacent critical points can be determined simply by finding the sign of $f'(x)$ for any *one* x in this interval.

Our sketch of the graph of $f(x) = x^3 - x$ contains sufficient information to allow us to say with confidence that $-\sqrt{1/3}$ is a local maximum point, and $\sqrt{1/3}$ a local minimum point. In fact, we can give a general scheme for

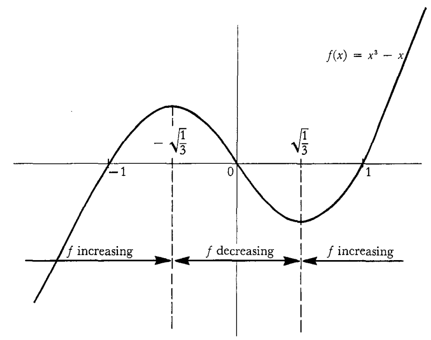


FIGURE 15

deciding whether a critical point is a local maximum point, a local minimum point, or neither (Figure 16):

(1) if $f' > 0$ in some interval to the left of x and $f' < 0$ in some interval to the right of x, then x is a local maximum point.
(2) if $f' < 0$ in some interval to the left of x and $f' > 0$ in some interval to the right of x, then x is a local minimum point.
(3) if f' has the same sign in some interval to the left of x as it has in some interval to the right, then x is neither a local maximum nor a local minimum point.

(There is no point in memorizing these rules—you can always draw the pictures yourself.)

The polynomial functions can all be analyzed in this way, and it is even possible to describe the general form of the graph of such functions. To begin,

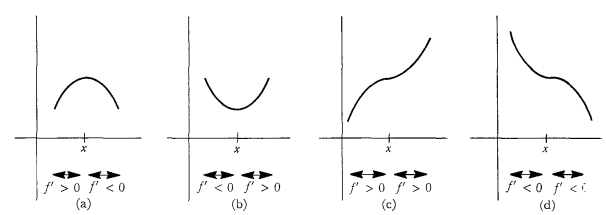


FIGURE 16

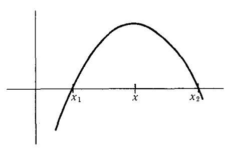


FIGURE 17

we need a result already mentioned in Problem 3-7: If

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0,$$

then f has at most n "roots," i.e., there are at most n numbers x such that $f(x) = 0$. Although this is really an algebraic theorem, calculus can be used to give an easy proof. Notice that if x_1 and x_2 are roots of f (Figure 17), so that $f(x_1) = f(x_2) = 0$, then by Rolle's Theorem there is a number x between x_1 and x_2 such that $f'(x) = 0$. This means that if f has k different roots $x_1 < x_2 < \cdots < x_k$, then f' has at least $k - 1$ different roots: one between x_1 and x_2, one between x_2 and x_3, etc. It is now easy to prove by induction that a polynomial function

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$$

has at most n roots: The statement is surely true for $n = 1$, and if we assume that it is true for n, then the polynomial

$$g(x) = b_{n+1}x^{n+1} + b_nx^n + \cdots + b_0$$

could not have more than $n + 1$ roots, since if it did, g' would have more than n roots.

With this information it is not hard to describe the graph of

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0.$$

The derivative, being a polynomial function of degree $n - 1$, has at most $n - 1$ roots. Therefore f has at most $n - 1$ critical points. Of course, a critical point is not necessarily a local maximum or minimum point, but at any rate, if a and b are adjacent critical points of f, then f' will remain either positive or negative on (a, b), since f' is continuous; consequently, f will be either increasing or decreasing on (a, b). Thus f has at most n regions of decrease or increase.

As a specific example, consider the function

$$f(x) = x^4 - 2x^2.$$

Since

$$f'(x) = 4x^3 - 4x = 4x(x - 1)(x + 1),$$

the critical points of f are -1, 0, and 1, and

$$f(-1) = -1,$$
$$f(0) = 0,$$
$$f(1) = -1.$$

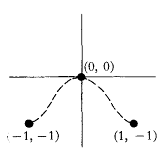


FIGURE 18

The behavior of f on the intervals between the critical points can be determined by one of the methods mentioned before. In particular, we could determine the sign of f' on these intervals simply by examining the formula for $f'(x)$. On the other hand, from the three critical values alone we can see (Figure 18) that f increases on $(-1, 0)$ and decreases on $(0, 1)$. To determine the sign of f' on $(-\infty, -1)$ and $(1, \infty)$ we can compute

$$f'(-2) = 4 \cdot (-2)^3 - 4 \cdot (-2) = -24,$$
$$f'(2) = 4 \cdot 2^3 - 4 \cdot 2 = 24,$$

and conclude that f is decreasing on $(-\infty, -1)$ and increasing on $(1, \infty)$. These conclusions also follow from the fact that $f(x)$ is large for large x and for large negative x.

We can already produce a good sketch of the graph; two other pieces of information provide the finishing touches (Figure 19). First, it is easy to determine that $f(x) = 0$ for $x = 0, \pm\sqrt{2}$; second, it is clear that f is even, $f(x) = f(-x)$, so the graph is symmetric with respect to the vertical axis. The function $f(x) = x^3 - x$, already sketched in Figure 15, is odd, $f(x) = -f(-x)$, and is consequently symmetric with respect to the origin. Half the work of graph sketching may be saved by noticing these things in the beginning.

Several problems in this and succeeding chapters ask you to sketch the graphs of functions. In each case you should determine

(1) the critical points of f,
(2) the value of f at the critical points,
(3) the sign of f' in the regions between critical points (if this is not already clear),
(4) the numbers x such that $f(x) = 0$ (if possible),
(5) the behavior of $f(x)$ as x becomes large or large negative (if possible).

Finally, bear in mind that a quick check, to see whether the function is odd or even, may save a lot of work.

This sort of analysis, if performed with care, will usually reveal the basic shape of the graph, but sometimes there are special features which require a little more thought. It is impossible to anticipate all of these, but one piece of information is often very important. If f is not defined at certain points (for example, if f is a rational function whose denominator vanishes at some points), then the behavior of f near these points should be determined.

For example, consider the function

$$f(x) = \frac{x^2 - 2x + 2}{x - 1},$$

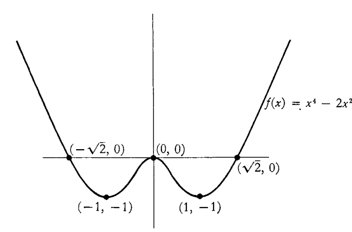


FIGURE 19

which is not defined at 1. We have

$$\begin{aligned} f'(x) &= \frac{(x-1)(2x-2) - (x^2-2x+2)}{(x-1)^2} \\ &= \frac{x(x-2)}{(x-1)^2}. \end{aligned}$$

Thus

(1) the critical points of f are 0, 2.

Moreover,

(2) $f(0) = -2,$
$f(2) = 2.$

Because f is not defined on the whole interval $(0, 2)$, the sign of f' must be determined separately on the intervals $(0, 1)$ and $(1, 2)$, as well as on the intervals $(-\infty, 0)$ and $(2, \infty)$. We can do this by picking particular points in each of these intervals, or simply by staring hard at the formula for f'. Either way we find that

$$\begin{aligned} (3)\ f'(x) &> 0 \quad \text{if} \quad x < 0, \\ f'(x) &< 0 \quad \text{if} \quad 0 < x < 1, \\ f'(x) &< 0 \quad \text{if} \quad 1 < x < 2, \\ f'(x) &> 0 \quad \text{if} \quad 2 < x. \end{aligned}$$

Finally, we must determine the behavior of $f(x)$ as x becomes large or large negative, as well as when x approaches 1 (this information will also give us another way to determine the regions on which f increases and decreases). To examine the behavior as x becomes large we write

$$\frac{x^2-2x+2}{x-1} = x - 1 + \frac{1}{x-1};$$

clearly $f(x)$ is close to $x - 1$ (and slightly larger) when x is large, and $f(x)$ is close to $x - 1$ (but slightly smaller) when x is large negative. The behavior of f near 1 is also easy to determine; since

$$\lim_{x\to 1} (x^2 - 2x + 2) = 1 \neq 0,$$

the fraction

$$\frac{x^2-2x+2}{x-1}$$

becomes large as x approaches 1 from above and large negative as x approaches 1 from below.

All this information may seem a bit overwhelming, but there is only one way that it can be pieced together (Figure 20); be sure that you can account for each feature of the graph.

When this sketch has been completed, we might note that it looks like the

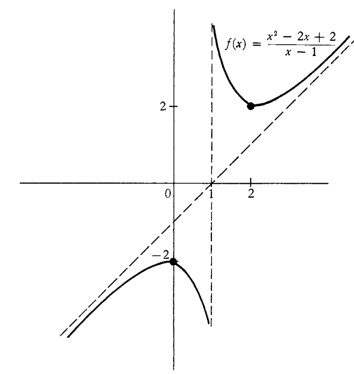


FIGURE 20

graph of an odd function shoved over 1 unit, and the expression

$$\frac{x^2-2x+2}{x-1} = \frac{(x-1)^2+1}{x-1}$$

shows that this is indeed the case. However, this is one of those special features which should be investigated only after you have used the other information to get a good idea of the appearance of the graph.

Although the location of local maxima and minima of a function is always revealed by a detailed sketch of its graph, it is usually unnecessary to do so much work. There is a popular test for local maxima and minima which depends on the behavior of the function only at its critical points.

THEOREM 5 Suppose $f'(a) = 0$. If $f''(a) > 0$, then f has a local minimum at a; if $f''(a) < 0$, then f has a local maximum at a.

PROOF By definition,

$$f''(a) = \lim_{h\to 0} \frac{f'(a+h) - f'(a)}{h}.$$

Since $f'(a) = 0$, this can be written

$$f''(a) = \lim_{h\to 0} \frac{f'(a+h)}{h}.$$

Suppose now that $f''(a) > 0$. Then $f'(a+h)/h$ must be positive for sufficiently small h. Therefore:

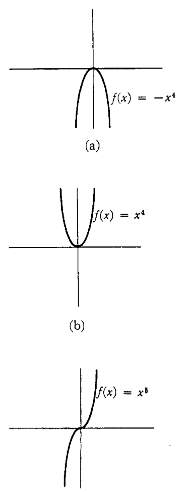


FIGURE 21

$$f'(a+h) \text{ must be positive for sufficiently small } h > 0$$
$$\text{and } f'(a+h) \text{ must be negative for sufficiently small } h < 0.$$

This means (Corollary 3) that f is increasing in some interval to the right of a and f is decreasing in some interval to the left of a. Consequently, f has a local minimum at a.

The proof for the case $f''(a) < 0$ is similar. ▮

Theorem 5 may be applied to the function $f(x) = x^3 - x$, which has already been considered. We have

$$f'(x) = 3x^2 - 1$$
$$f''(x) = 6x.$$

At the critical points, $-\sqrt{1/3}$ and $\sqrt{1/3}$, we have

$$f''(-\sqrt{1/3}) = -6\sqrt{1/3} < 0,$$
$$f''(\sqrt{1/3}) = 6\sqrt{1/3} > 0.$$

Consequently, $-\sqrt{1/3}$ is a local maximum point and $\sqrt{1/3}$ is a local minimum point.

Although Theorem 5 will be found quite useful for polynomial functions, the second derivative of many functions is so complicated that it is easier to consider the sign of the first derivative. Moreover, if a is a critical point of f it may happen that $f''(a) = 0$. In this case, Theorem 5 provides no information: it is possible that a is a local maximum point, a local minimum point, or neither, as shown (Figure 21) by the functions

$$f(x) = -x^4, \quad f(x) = x^4, \quad f(x) = x^5;$$

in each case $f'(0) = f''(0) = 0$, but 0 is a local maximum point for the first, a local minimum point for the second, and neither a local maximum nor minimum point for the third. This point will be pursued further in Part IV.

It is interesting to note that Theorem 5 automatically proves a partial converse of itself.

THEOREM 6 Suppose $f''(a)$ exists. If f has a local minimum at a, then $f''(a) \geq 0$; if f has a local maximum at a, then $f''(a) \leq 0$.

PROOF Suppose f has a local minimum at a. If $f''(a) < 0$, then f would also have a local maximum at a, by Theorem 5. Thus f would be constant in some interval containing a, so that $f''(a) = 0$, a contradiction. Thus we must have $f''(a) \geq 0$.

The case of a local maximum is handled similarly. ▮

(This partial converse to Theorem 5 is the best we can hope for: the $\geq$ and $\leq$ signs cannot be replaced by $>$ and $<$, as shown by the functions $f(x) = x^4$ and $f(x) = -x^4$.)

The remainder of this chapter deals, not with graph sketching, or maxima and minima, but with three consequences of the Mean Value Theorem. The

first is a simple, but very beautiful, theorem which plays an important role in Chapter 15, and which also sheds light on many examples which have occurred in previous chapters.

THEOREM 7 Suppose that f is continuous at a, and that $f'(x)$ exists for all x in some interval containing a, except perhaps for $x = a$. Suppose, moreover, that $\lim_{x \to a} f'(x)$ exists. Then $f'(a)$ also exists, and

$$f'(a) = \lim_{x \to a} f'(x).$$

PROOF By definition,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

For sufficiently small $h > 0$ the function f will be continuous on $[a, a+h]$ and differentiable on $(a, a+h)$ (a similar assertion holds for sufficiently small $h < 0$). By the Mean Value Theorem there is a number α_h in $(a, a+h)$ such that

$$\frac{f(a+h) - f(a)}{h} = f'(\alpha_h).$$

Now α_h approaches a as h approaches 0, because α_h is in $(a, a+h)$; since $\lim_{x \to a} f'(x)$ exists, it follows that

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} f'(\alpha_h) = \lim_{x \to a} f'(x).$$

(It is a good idea to supply a rigorous ε-δ argument for this final step, which we have treated somewhat informally.) ▮

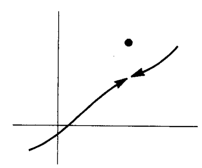


FIGURE 22

Even if f is an everywhere differentiable function, it is still possible for f' to be discontinuous. This happens, for example, if

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

According to Theorem 7, however, the graph of f' can never exhibit a discontinuity of the type shown in Figure 22. Problem 55 outlines the proof of another beautiful theorem which gives further information about the function f', and Problem 56 uses this result to strengthen Theorem 7.

The next theorem, a generalization of the Mean Value Theorem, is of interest mainly because of its applications.

THEOREM 8 (THE CAUCHY MEAN VALUE THEOREM) If f and g are continuous on $[a, b]$ and differentiable on (a, b), then there is a number x in (a, b) such that

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x).$$

(If $g(b) \neq g(a)$, and $g'(x) \neq 0$, this equation can be written

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x)}{g'(x)}.$$

Notice that if $g(x) = x$ for all x, then $g'(x) = 1$, and we obtain the Mean Value Theorem. On the other hand, applying the Mean Value Theorem to f and g separately, we find that there are x and y in (a, b) with

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x)}{g'(y)};$$

but there is no guarantee that the x and y found in this way will be equal. These remarks may suggest that the Cauchy Mean Value Theorem will be quite difficult to prove, but actually the simplest of tricks suffices.)

PROOF Let

$$h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)].$$

Then h is continuous on $[a, b]$, differentiable on (a, b), and

$$h(a) = f(a)g(b) - g(a)f(b) = h(b).$$

It follows from Rolle's Theorem that $h'(x) = 0$ for some x in (a, b), which means that

$$0 = f'(x)[g(b) - g(a)] - g'(x)[f(b) - (a)]. \blacksquare$$

The Cauchy Mean Value Theorem is the basic tool needed to prove a theorem which facilitates evaluation of limits of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)},$$

when

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0.$$

In this case, Theorem 5-3 is of no use. Every derivative is a limit of this form, and computing derivatives frequently requires a great deal of work. If some derivatives are known, however, many limits of this form can now be evaluated easily.

THEOREM 9 (L'HÔPITAL'S RULE) Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0,$$

and suppose also that $\lim_{x \to a} f'(x)/g'(x)$ exists. Then $\lim_{x \to a} f(x)/g(x)$ exists, and

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

(Notice that Theorem 7 is a special case.)

PROOF The hypothesis that $\lim_{x \to a} f'(x)/g'(x)$ exists contains two implicit assumptions:

(1) there is an interval $(a - \delta, a + \delta)$ such that $f'(x)$ and $g'(x)$ exist for all x in $(a - \delta, a + \delta)$ except, perhaps, for $x = a$,
(2) in this interval $g'(x) \neq 0$ with, once again, the possible exception of $x = a$.

On the other hand, f and g are not even assumed to be defined at a. If we define $f(a) = g(a) = 0$ (changing the previous values of $f(a)$ and $g(a)$, if necessary), then f and g are continuous at a. If $a < x < a + \delta$, then the Mean Value Theorem and the Cauchy Mean Value Theorem apply to f and g on the interval $[a, x]$ (and a similar statement holds for $a - \delta < x < a$). First applying the Mean Value Theorem to g, we see that $g(x) \neq 0$, for if $g(x) = 0$ there would be some x_1 in (a, x) with $g'(x_1) = 0$, contradicting (2). Now applying the Cauchy Mean Value Theorem to f and g, we see that there is a number α_x in (a, x) such that

$$[f(x) - 0]g'(\alpha_x) = [g(x) - 0]f'(\alpha_x)$$

or

$$\frac{f(x)}{g(x)} = \frac{f'(\alpha_x)}{g'(\alpha_x)}.$$

Now α_x approaches a as x approaches a, because α_x is in (a, x); since $\lim_{y \to a} f'(y)/g'(y)$ exists, it follows that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(\alpha_x)}{g'(\alpha_x)} = \lim_{y \to a} \frac{f'(y)}{g'(y)}.$$

(Once again, the reader is invited to supply the details of this part of the argument.) $\blacksquare$

PROBLEMS

1. For each of the following functions, find the maximum and minimum values on the indicated intervals, by finding the points in the interval where the derivative is 0, and comparing the values at these points with the values at the end points.

(i) $f(x) = x^3 - x^2 - 8x + 1$ on $[-2, 2]$.
(ii) $f(x) = x^5 + x + 1$ on $[-1, 1]$.
(iii) $f(x) = 3x^4 - 8x^3 + 6x^2$ on $[-\frac{1}{2}, \frac{1}{2}]$.
(iv) $f(x) = \dfrac{1}{x^5 + x + 1}$ on $[-\frac{1}{2}, 1]$.
(v) $f(x) = \dfrac{x + 1}{x^2 + 1}$ on $[-1, \frac{1}{2}]$.
(vi) $f(x) = \dfrac{x}{x^2 - 1}$ on $[0, 5]$.

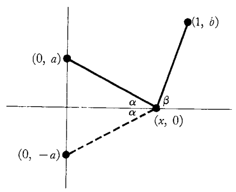


FIGURE 23

Surface area is the sum of these areas

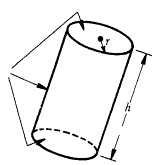


FIGURE 24

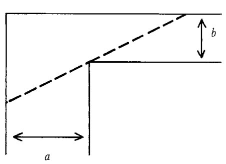


FIGURE 25

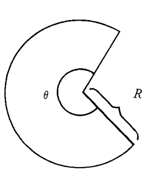


FIGURE 26

6. (a) Let (x_0, y_0) be a point of the plane, and let L be the graph of the function $f(x) = mx + b$. Find the point $\bar{x}$ such that the distance from (x_0, y_0) to $(\bar{x}, f(\bar{x}))$ is smallest. [Notice that minimizing this distance is the same as minimizing its square. This may simplify the computations somewhat.]

(b) Also find $\bar{x}$ by noting that the line from (x_0, y_0) to $(\bar{x}, f(\bar{x}))$ is perpendicular to L.

(c) Find the distance from (x_0, y_0) to L, i.e., the distance from (x_0, y_0) to $(\bar{x}, f(\bar{x}))$. [It will make the computations easier if you first assume that $b = 0$; then apply the result to the graph of $f(x) - mx$ and the point $(x_0, y_0 - b)$.] Compare with Problem 4-22.

(d) Consider a straight line described by the equation $Ax + By + C = 0$ (Problem 4-7). Show that the distance from (x_0, y_0) to this line is $(Ax_0 + By_0 + C)/\sqrt{A^2 + B^2}$.

7. The previous Problem suggests the following question: What is the relationship between the critical points of f and those of f^2?

8. A straight line is drawn from the point $(0, a)$ to the horizontal axis, and then back to $(1, b)$, as in Figure 23. Prove that the total length is shortest when the angles α and β are equal. (Naturally you must bring a function into the picture: express the length in terms of x, where $(x, 0)$ is the point on the horizontal axis. The dashed line in Figure 23 suggests an alternative geometric proof; in either case the problem can be solved without actually finding the point $(x, 0)$.)

9. Prove that of all rectangles with given perimeter, the square has the greatest area.

10. Find, among all right circular cylinders of fixed volume V, the one with smallest surface area (counting the areas of the faces at top and bottom, as in Figure 24).

11. A right triangle with hypotenuse of length a is rotated about one of its legs to generate a right circular cone. Find the greatest possible volume of such a cone.

12. Two hallways, of widths a and b, meet at right angles (Figure 25). What is the greatest possible length of a ladder which can be carried horizontally around the corner?

13. A garden is to be designed in the shape of a circular sector (Figure 26), with radius R and central angle θ. The garden is to have a fixed area A. For what value of R and θ (in radians) will the length of the fencing around the perimeter be minimized?

14. Show that the sum of a number and its reciprocal is at least 2.

15. Find the trapezoid of largest area that can be inscribed in a semicircle of radius a, with one base lying along the diameter.

GreatHearts Irving

Remote Learning Packet

NB: Please keep all work produced this week. Details regarding how to turn in this work will be forthcoming.

April 27 - May 1, 2020
Course: 11 Drama
Teacher(s): Mrs. Jimenez (margaret.cousino@greatheartsirving.org)

Weekly Plan:

Lines should be mastered by the end of this week!

Monday, April 27
- [ ] Practice lines for 20 minutes

Tuesday, April 28
- [ ] Practice lines for 20 minutes
- [ ] 11am-12pm: Zoom Rehearsal - Act 1

(only actors with lines in Act 1 and Student Director)

Wednesday, April 29
- [ ] Practice lines for 20 minutes

Thursday, April 30
- [ ] Practice lines for 20 minutes

Friday, May 1
- [ ] Practice lines for 20 minutes

Statement of Academic Honesty

I affirm that the work completed from the packet is mine and that I completed it independently.

I affirm that, to the best of my knowledge, my child completed this work independently

Student Signature

Parent Signature

Monday, April 27

- Practice lines for 20 minutes. Every day you should review the lines you have already mastered without looking, then focus on a new scene. If you are a lead and have many lines, choose one scene to review and a new one to work on. Every day you should review a different scene you already have memorized and work for as many days as necessary to master the new one. Record your time on the sheet.

Line memorizing strategies:

- Recite your lines OUT LOUD. Practice them like you will say them on stage – projecting, appropriate speed and emotion, etc. Ask yourself why your character is saying what he/she says and that will help you interpret how to say the line.
- Run your lines with a friend or family member. They should read the lines of the other characters in your scenes while you practice your lines from memory.
- Practice your lines in front of a mirror—the bigger the better! Watch yourself—your facial expressions, how you move, stand, etc.—to be aware of how you look while saying your lines.
- Record yourself saying your lines and listen to the audio (even better if you record your cues!)
- Write out your lines by hand (especially if you have a long speech, it is helpful to get it into your memory through writing it out multiple times).
- KNOW YOUR CUES! What line or action comes before you speak?
- Run through the parts of the scenes in which you do not speak—what is your character doing during those parts of the play?
- After spending a period of time going over your lines, take a walk or a nap
- REMEMBER: Consistent practice is the key to success!

ALL LINES MUST BE MEMORIZED BY THE END OF THIS WEEK!

Tuesday, April 28

- Practice lines for 20 minutes according to Monday's directions. Record your time on the sheet.
- **If you are in Act 1 or the Student Director, join the Zoom meeting for online rehearsal from 11am-12pm.** This hour will count as your rehearsal time for the week (unless you are woefully behind memorizing lines). Record that you attended the Zoom rehearsal in your sheet to turn in. If you're not in Act 1, you don't need to attend; just practice your lines like usual.
- Zoom link: https://zoom.us/j/94257807722?pwd=NDNHWFltclNkZnhCYS9QQzlPWDI3UT09

Wednesday, April 29

- Practice lines for 20 minutes according to Monday's directions. Record your time on the sheet.

Thursday, April 30

- Practice lines for 20 minutes according to Monday's directions. Record your time on the sheet.t.

Friday, May 1

- Practice lines for 20 minutes according to Monday's directions. Record your time on the sheet.

Drama Weekly Line Memorization

Name: **Week: 4/27-5/3**

Day:	Minutes practiced:
Monday	
Tuesday	
Wednesday	
Thursday	
Friday	
Saturday	
Sunday	

Minimum time: 20 mins/day, 5 days/week

I verify that this is a true and accurate account of the time I have spent memorizing my lines this past week.

Signature: **Date:**

GreatHearts Irving

Remote Learning Packet

April 27 - May 1, 2020
Course: 11 Greek
Teacher(s): Miss Salinas annie.salinas@greatheartsirving.org

Weekly Plan:

Monday, April 27
☐ If you haven't already, accept the invitation to join the Google Classroom for 11th Greek!
☐ Worksheet: Counting; Exercise 8ι
☐ 40 reps of exercise (20 cardinal, 20 ordinal)

Tuesday, April 28
☐Worksheet: Exercise 8λ
☐ 30 reps of exercise (cardinal)

Wednesday, April 29
☐ Worksheet: 1st half of of ὁ Ὀδυσσευς και ὁ Αἰολος
☐ 30 reps of exercise (ordinal)

Thursday, April 30
☐ Worksheet: 2nd half of ὁ Ὀδυσσευς και ὁ Αἰολος
☐ 40 reps of exercise (cardinal)

Friday, May 1
☐ Come to Office Hours at 10:30am! (link available in the stream of our Google Classroom)
☐ Worksheet: Exercise 8θ
☐ 40 reps of exercise (ordinal)

A note on Google Classroom:

At time of writing this note on Wednesday, 22 April, **just over half of the students in this class have accepted the invitation to join the Google Classroom**. Don't languish in the dark! If you are having trouble accessing it, please reach out to me or Ms. Weatherton, our registrar, who can help get you set up. This week (Monday, April 27, when you are reading this), our work is completely posted on Google Classroom in handy-dandy daily assignments you can complete without ever having to print and scan anything, along with links to Friday's Office Hours. **If you are one of the nine students who has not done so, *please log in to your Google Classroom as soon as you have access to a computer and accept the invitation to join 11th Greek!***

A note on the second exercise (heh, literally) of each day:

This week, one of my favorite youtube channels that makes content related to teaching classical languages (I'm aware of my critical level of nerdery) posted a video with tips on how to master counting in foreign languages. What luck! His pro-tip is this: if you use the numbers of your target language to count reps of exercise, your brain is much more likely to latch on to them. (Pending permission, I'll link the video in our Google Classroom. He's better at explaining things than I am.) **Our second assignment of each day this week is this: do a number of reps of your choice of the following: push-ups, sit-ups, squats, step-ups, calf raises, crunches, plans (in seconds), or wall-sits (in seconds) and count them in Greek.** If you need to make a paper copy of the numbers to hold or put in front of you for the first few days, do!

A note on sanity levels:

How are you? Are you staying sane? Are you either bored and unchallenged or overwhelmed? Are the days of the week losing all meaning? Is time losing all meaning? Is *meaning* losing all meaning? Same. I care about your sanity, especially as Greek relates to it. **If you would like more of a challenge or if the workload is getting to be too much, reach out and let me know.** I'm creating these lessons week by week, so I have the flexibility to adapt subsequent plans according to how the class is doing. So far it seems alright, but if you have feedback, shoot it to me. Hey, you know a good way to send me feedback? *Logging on to your computer and emailing me! While you're there, accept the invitation to join the Google Classroom! (Okay, I'll stop...I realize I may be a cause of your potentially low sanity levels.)*

Monday, April 27

- ➔ Complete Monday's one-page worksheet, which includes counting and Exercise 8ι.
- ➔ Today, do forty reps of your exercise of choice, counting twice to δεκα (cardinal numbers) and twice to δεκατος (ordinal numbers).

Tuesday, April 28

- ➔ Complete Tuesday's one-page worksheet, which includes Exercise 8λ. ***This worksheet is best done on Google Classroom.***
- ➔ Today, do thirty reps of your exercise of choice, counting thrice to δεκα (cardinal numbers).

Wednesday, April 29

- ➔ Complete Wednesday's two-page worksheet, which includes the first half of a story about Odysseus. ***This worksheet is best done on Google Classroom.***
- ➔ Today, do thirty reps of your exercise of choice, counting thrice to δεκατος (ordinal numbers).

Thursday, April 30

- ➔ Complete Thursday's two-page worksheet, which includes the second half of a story about Odysseus. ***This worksheet is best done on Google Classroom.***
- ➔ Today, do forty reps of your exercise of choice, counting four times to δεκα (cardinal numbers).

Friday, May 1

- ➔ Come to Office Hours at 10:30am! Link available in the stream of our Google Classroom. Ask questions, give me feedback on your workload and the best format for worksheets, or at least connect on a human level with someone you're not quarantined with.
- ➔ Complete Friday's two-page worksheet, which includes Exercise 8θ. ***This worksheet is best done on Google Classroom.***
- ➔ Today, do forty reps of your exercise of choice, counting four times to δεκατος (ordinal numbers).

Monday

Counting to δεκα / Exercise 8ι

Write the cardinal and ordinal numbers in Greek and English here.

CARDINAL	ORDINAL
1.	1.
2.	2.
3.	3.
4. τετταρες (m./f.), τετταρα (n.) - four	4. τεταρος, τεταρη, τεταρον - fourth
5.	5.
6.	6.
7.	7.
8.	8.
9.	9.
10.	10.

Exercise 8ι (from workbook)

Select the English equivalent from the word bank above and write it on the line in front of each Greek numerical adjective.

1. ________________ τεταρτος
2. ________________ ἑπτα
3. ________________ ἐνατος
4. ________________ ἑξ
5. ________________ δεκατος
6. ________________ ὀκτω
7. ________________ δευτερος
8. ________________ τρεις
9. ________________ δεκα
10. ________________ πεμπτος
11. ________________ ὀγδοος
12. ________________ πεντε
13. ________________ ἑν
14. ________________ τριτος
15. ________________ πρωτος
16. ________________ εἷς
17. ________________ δυο
18. ________________ ἑβδομος
19. ________________ ἑκτος
20. ________________ τετταρες
21. ________________ ἐννεα
22. ________________ μια

Tuesday

Exercise 8λ

This worksheet is best done on Google Classroom. *On the space to the right of each sentence, translate the underlined phrase. Then, circle the type of time expression used.*

1. We worked <u>τρεις ἡμερας.</u> ________________________

   gen. of time within which   dat. of time when   acc. duration of time

2. It all took place <u>τριῶν ἡμερῶν</u>. ________________________

   gen. of time within which   dat. of time when   acc. duration of time

3. She arrived <u>τῇ τριτῇ ἡμερᾳ</u>. ________________________

   gen. of time within which   dat. of time when   acc. duration of time

4. They had the work finished <u>πεντε ἡμερῶν</u>. ________________________

   gen. of time within which   dat. of time when   acc. duration of time

5. <u>πασαν την ἡμεραν</u> the farmer was working. ________________________

   gen. of time within which   dat. of time when   acc. duration of time

6. The athlete received a prize <u>τῇ πεμπη ἡμερᾳ</u>. ________________________

   gen. of time within which   dat. of time when   acc. duration of time

7. <u>τῇ ὑστεραιᾳ</u> they left for the harbor. ________________________

   gen. of time within which   dat. of time when   acc. duration of time

8. They traveled <u>πολυν χρονον</u>. ________________________

   gen. of time within which   dat. of time when   acc. duration of time

Wednesday

ὁ Ὀδυσσευς και ὁ Αἰολος - ἡμισυ πρωτον (“1st half”)

This worksheet is best done on Google Classroom. *Read the following passage and answer the comprehension questions.*

Odysseus tells how he sailed on to the island of Aeolus, king of the winds, and almost reached home:

ἐπει δε ἐκ του ἀντρου (*the cave*) του Κυκλωπος ἐκφευγομεν, ἐπανερχομεθα (*we return*) ταχεως προς τους ἑταιρους. οἱ δε, ἐπει ἡμας ὁρωσιν, χαιρουσιν. τῇ δ’ὑστεραιᾳ κελευω αὐτους εἰς την ναυν αὐθις εἰσβαινειν. οὑτως οὐν ἀποπλεομεν.

1. What do Odysseus and his men do when they escape from the cave of the Cyclops?

___

___

2. What does Odysseus order his men to do the next day?

___

___

3. τῇ δ’ὑστεραιᾳ is which of the following?
   gen. of time within which | dat. of time when | acc. duration of time

δι’ὀλιγου δε εἰς νησον Αἰολιαν (*Aeolia - the island of Aeolus, king of the winds*) ἀφικνουμεθα. ἐκει δε οἰκει ὁ Αἰολος, βασιλευς των ἀνεμων (*of the winds*). ἡμας δε εὐμενως (*kindly*) δεχομενος πολυν χρονον ξενιζει (*entertains*). ἐπει δε ἐγω κελευω αὐτον ἡμας ἀποπεμπειν, παρεχει μοι ἀσκον (*bag*) τινα, εἰς ὅν (*which*) παντας τους ἀνεμους καταδει (*he ties up*) πλην (+gen, *except*) ἑνος, Ζεφυρου πραου (*gentle*).

4. Where do Odysseus and his men arrive next?

5. How long do Odysseus and his men stay with Aeolus?

6. What does Aeolus give Odysseus at his departure?

7. What wind was not in the bag?

Thursday

ὁ Ὀδυσσευς και ὁ Αἰολος - ἡμισυ δευτερον (“2nd half”)

This worksheet is best done on Google Classroom. *Read the following passage and answer the comprehension questions.*

ἐννεα μεν οὐν ἡμερας πλεομεν, τῇ δε δεκατῇ ὁρωμεν την πατριδα γην (*our fatherland*). ἐνταυθα δη ἐγω καθευδω. οἱ δε ἑταιροι, ἐπει ὁρωσι με καθευδοντα (*sleeping*), οὑτω λεγουσιν. “τι ἐν τῷ ἀσκῷ (*bag*) ἐνεστιν; πολυς δηπου (*surely*) χρυσος (*gold*) ἐνεστιν, πολυ τε ἀργυριον (*silver*), δωρα (*gifts*) του Αἰολου. ἀγετε δη (*come on!*), λυετε τον ἀσκον και τον χρυσον αἱρειτε (*take*).”

1. How long do Odysseus and his men sail for?

2. ἐννεα...ἡμερας is which of the following?
   gen. of time within which — dat. of time when — acc. duration of time

3. τῇ δε δεκατῇ is which of the following?
   gen. of time within which — dat. of time when — acc. duration of time

4. When they come within sight of their fatherland, what does Odysseus do?

5. What do his comrades think is in the bag?

ἐπει δε λυουσι τον ἀσκον, εὐθυς (*at once*) ἐκπετονται (*fly out*) παντες οἱ ἀνεμοι και χειμωνα δεινον ποιουσι και την ναυν ἀπο της πατριδος γης ἀπελαυνουσιν (*they expel/push away)*. ἐγω δε ἐγειρομαι (*wake myself up*) και γιγνώσκω τι γιγνεται (*happens*). ἀθυμω (*I despair*) οὐν και βουλομαι ῥιπτειν (*to throw*) ἐμαυτον (*myself*) εἰς την θαλατταν. οἱ δε ἑταιροι σῳζουσι με. οὑτως οὐν οἱ ἀνεμοι ἡμας εἰς την του Αἰολου νησον παλιν (*again*) φερουσιν.

6. What happens when the men open the bag?

___

___

7. How does Odysseus react when he wakes up?

___

___

8. Why doesn't Odysseus carry out his plan of despair?

___

___

9. Where do the winds carry the ship at last?

___

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Friday

Exercise 8θ

This worksheet is best done on Google Classroom. *Answer the questions and translate the following sentences, which continue the story from Odysseus' perspective.*

1. ἐπει εἰς την νησον ἀφικνουμεθα, προς τον του Αἰολου οἰκον ἐρχομαι.

   a. Which is correct about ἀφικνουμεθα?

   2nd person sg: you arrive   1st person pl: we arrive   2nd person pl: y'all arrive

   b. What case is του Αἰολου, and what keyword thus goes before it when translating?

   gen: of   dat: in, on   acc: no keyword

Translation: ____

2. ὁ δε, ἐπει ὁρᾷ με, μαλα θαυμαζει και, "τι πασχεις;" φησιν, "τι αὐθις παρει;"

   a. Are the verbs ὁρᾷ, θαυμαζει, and πασχεις in the active or middle voice?

   active   middle

   b. Which punctuation mark is ; in Greek equivalent to in English?

   -    :   ?   !   ,   .

Translation: ____

3. ἐγω δε ἀποκρινομαι, "οἱ ἑταιροι αἴτιοι (*to blame*) εἰσιν. τους γαρ ἀνεμους (*the winds*) ἐλυσαν (*they loosed*). ἀλλα βοηθει ἑμιν, ὠ φιλε."

   a. Which of the following does NOT describe Odysseus' tone?

   suppliant/begging   responsible/mature   blaming/selfish

   b. What little thing does Odysseus do in the last sentence to appeal to Aeolus and attempt to make him more willing to help?

   ____

Translation: ____

4. ὁ δε Αἰολος, "ἀπιτε (go away) ταχεως," φησιν, "ἀπο της νησου. οὐ γαρ δυνατον ἐστιν ὑμιν βοηθειν. οἱ γαρ θεοι δηπου (surely) μισουσιν (hate) ὑμας."

   a. Does Aeolus want to help Odysseus or not?

   of course, yes!   not at all

   b. What mood is ἀπιτε?

   indicative   imperative   infinitive   subjunctive

Translation: ____

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Remote Learning Packet

NB: Please keep all work produced this week. Details regarding how to turn in this work will be forthcoming.

April 27 - May 1, 2020

Course: 11 Humane Letters

Teacher(s): Mr. Brandolini david.brandolini@greatheartsirving.org

Mr. Mercer andrew.mercer@greatheartsirving.org

Welcome to Week 5! In the hopes of providing the greatest opportunity to read *Hamlet* with the leisure and appreciation it deserves, we will be scaling back the written work of this week to primarily focus on enjoying this timeless cornerstone of the Western Tradition. Daily comments for each reading are included, kept deliberately brief to serve as signposting for major themes/moments to keep an eye out for before taking on the day's reading. For this week, there will only be two reading quizzes, on Wednesday and Friday. We will revisit *Hamlet* for a deeper essay at the start of next week.

Weekly Plan:

Monday, April 27

☐ Read William Shakespeare, *Hamlet*, Act 1 (p. 3-69 of the Folger edition)

Tuesday, April 28

☐ Read *Hamlet*, Act 2 (p. 73-119)

Wednesday, April 29

☐ Complete Quiz on *Hamlet*, Acts 1&2

☐ Read *Hamlet*, Act 3 (p. 123-185)

Thursday, April 30

☐ Read *Hamlet*, Act 4 (p. 189-235)

Friday, May 1

☐ Read *Hamlet*, Act 5 (p. 239-287)

☐ Complete Quiz on *Hamlet*, Acts 3-5

Statement of Academic Honesty

I affirm that the work completed from the packet is mine and that I completed it independently.

I affirm that, to the best of my knowledge, my child completed this work independently

____

Student Signature

____

Parent Signature

Monday, April 27

Read William Shakespeare, *Hamlet*, Act 1 (p. 3-69 of the Folger edition)

At long last, we come to the end of our series of tragedies! It is fitting that we should end with the greatest of all, Shakespeare's *Hamlet*. If Sophocles' *Oedipus* exemplifies the perfect distillation of Aristotle's standards for tragic action, then *Hamlet* reinvigorates and elevates the movement to heights that have yet to be surpassed. In the most general sense, a tragic tale has three distinct parts:

1) a Fall that communicates the primary flaw, motive, or concern of the play;
2) a period of great stasis and suffering, or a Wandering;
3) a final Reconciliation, or a resolution of the play's action that leaves the hero or the cosmos of the story in a greater order than it started.

Thus far, we have seen that each play can tackle these stages to varying degrees of emphasis; in *Oresteia*, each play essentially serves as an emphasis on each respective stage, while also containing a complete movement within each individual work. In *Oedipus*, the vast majority of emphasis is in the first aspect, the Fall of the king and its causes. *Hamlet* is somewhat unusual in that it places primary emphasis on the latter two, leaving much of the action of the Fall to before the start of the drama. A key reason for this is that *Hamlet* is written as much in a Christian tradition as it is in a literary one; by this I mean that just as Homer, Aeschylus, and Sophocles innately concerned with the traditions and cosmology of their society, so too is *Hamlet* directly informed by a Christian world: it will be especially important to read the play with an eye towards the redemptive aspect of the play.

For the time being, however, simply begin! Avoid the temptation to over-analyze or to write off the sufferings and actions of the characters as merely confused or selfish; either extreme will distract you from simply opening yourself to the play's splendor.

Tuesday, April 28

Read *Hamlet*, Act 2 (p. 73-119)

"Something is rotten in the state of Denmark" says Marcellus in Act 1.4. As we continue onward into Act 2, we should hopefully begin to see that this is very much apparent. One of the prevailing themes of the play is infection, sickness of soul, and sin, the "vicious mole of nature" in men since birth (1.4.26); according to the Ghost in 1.5, the very cause of the descent of Denmark is due to his death, wherein "sleeping in my orchard, / A serpent stung me. So the whole ear of Denmark / is by a forged process of my death / rankly abused." (1.5.42-45). In this sense, the "Fall" of the drama is *the* Fall of Original Sin: Hamlet is plagued by the problem of sin and guilt, particularly as he is given the apparently impossible task of getting revenge for his father and remaining pure (much like Orestes' mandate from Apollo). Pay attention to imagery of infection and gardens.

As we continue into Act 2, remember Hamlet's plan to put on an "antic disposition": pay special attention to his different moods and behaviors, namely when performing before others, when confiding in a single person, or when alone and engaging in a soliloquy. One of the play's greatest ambiguities is to what extent Hamlet's antic disposition is an act. Hamlet very often discusses or emphasizes knowledge and ignorance: why is he so concerned with these, and do they indicate his tragic *hamartia* in any way?

Wednesday, April 29

Complete Quiz on *Hamlet*, Acts 1&2
Read *Hamlet*, Act 3 (p. 123-185)

Hamlet's famous reflection on the nature of suffering: what is it that paralyzes him so? Act 2 revealed that he has waited two months since the Ghost's command. Notice that Hamlet does not ask whether "To live", but rather "To be": his concern is existential, pondering whether man ought to exist at all, and is not merely born out of personal depression. His conversation with Ophelia following this soliloquy is also revealing of his assessment of human nature. Why does he speak to Ophelia the way he does? Is he trying to tell her something? How could these opening scenes be related to his concern over knowledge vs. ignorance?

Pay attention to how Hamlet reacts when approaching Claudius: we see both at their lowest and most desperate here, albeit in a somewhat ironic situation. As you continue reading, remember this moment and consider the consequences of Hamlet's decision. His interaction with his mother, meanwhile, provides a stark contrast with our previous tragic heroes: Hamlet refuses to cross any "natural" boundaries when dealing with his mother, and instead seeks to help her by revealing the severity of her sin and the state of her soul to herself. Perhaps some good will come of Hamlet's antic disposition?

Thursday, April 30

Read *Hamlet*, Act 4 (p. 189-235)

Hamlet's disposition has clearly changed--what has caused this? Does he feel any remorse or regret for what he has done in Act 3? Has he become no better than Claudius himself? The state Hamlet leaves Denmark in under Claudius as he leaves for England is dire; Claudius will put the blame on Hamlet, but is this not also an indication of Claudius' own inability to rule well? Notice also the stark contrast in the ways Laertes and Ophelia react to their father's death. To what extent does Laertes' thirst for revenge serve as a foil to Hamlet's?

Friday, May 1

Read *Hamlet*, Act 5 (p. 239-287)
Complete Quiz on *Hamlet*, Acts 3-5

By the end, has Hamlet had any reconciliation, any change of heart? Like much of the play, it is ambiguous, but the most critical passage begins at line 5.2.233, where we see Hamlet answer the question posed earlier of whether or not "To be". Is this moment a despairing admission of defeat, or a surrendering himself to God's Providence? Think back to the theme of Denmark and Original Sin: has

Hamlet managed to reconcile his understanding of the problem of sin? The play leaves Denmark's fate uncertain: has the rot been purged? Has Hamlet saved his family and his kingdom by revealing their errors? Once you've read, take some time to see if you can answer some of these questions for yourself. Next week, we will begin with a short essay that will attempt to assess these developments and see how they synthesize with the traditional, Aristotelian understanding of the tragic action.

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Wednesday April 29
Reading Quiz: *Hamlet*, Acts 1&2

1. What or who is it that Barnardo, Marcellus, and Horatio encounter during the night watch in scene 1?
    A. a wild beast
    B. a royal ghost
    C. a celestial sign
    D. a fleeing criminal

2. Which of the three characters is initially doubtful of their encounter?
    A. Horatio
    B. Barnardo
    C. Marcellus

3. What proposal does Horatio make at the end of scene 1?
    A. to make known to Hamlet what they encountered in the night
    B. to make known to Hamlet Ophelia's love for him
    C. to make known to Hamlet his uncle's misdeeds
    D. to make known to Hamlet a plot against his life

4. Identify the speaker: "Do not forever with thy vailed lids / Seek for thy noble father in the dust. / Thou know'st 'tis common; all that lives must die, / Passing through nature to eternity."
    A. King Claudius
    B. Polonius
    C. Hamlet
    D. Queen Gertrude

5. From where do Claudius and Gertrude try to persuade Hamlet from going?
    A. London
    B. Norway
    C. Wittenburg
    D. Paris

6. With regard to what does Laertes tell Ophelia to exercise utmost caution?
    A. Hamlet's love toward her
    B. Claudius's love toward her
    C. her father's counsel
    D. her own judgement

7. Identify the speaker: "This above all: to thine own self be true, / And it must follow, as the night the day, / Thou canst not then be false to any man."
    A. King Claudius
    B. Laertes
    C. Polonius
    D. Horatio

8. What does the spirit of young Hamlet's father reveal to him in the night?
   A. that the kingdom of Denmark will be overrun by Norsemen
   B. that Polonius was plotting to have Laertes become the next king
   C. that Gertrude was not his real mother
   D. that Claudius murdered him to take the throne and the queen

9. What does Hamlet demand of Horatio and Marcellus at the end of the first act?
   A. their allegiance to him in battle
   B. a vow of silence about what they saw that night
   C. money for him to travel to England
   D. their help solving a riddle

10. What does Ophelia report to her father in the opening scene of the second act?
   A. Hamlet's strange behavior toward her
   B. suspicious activity among Hamlet's friends
   C. that she's being courted by an English prince
   D. her ambition to become queen of Denmark

11. What does Claudius call on Rosencrantz and Guildenstern to do?
   A. discover the source of treachery in the palace
   B. find out why Hamlet has changed and try to cheer him up
   C. assassinate the prince of Norway
   D. spy on Laertes in France

12. Identify the speaker: "O, what a rogue and peasant slave am I!"
   A. Polonius
   B. Rosencrantz
   C. Guildenstern
   D. Hamlet

13. In two to three complete sentences, answer the following question: Does Hamlet have an optimistic or pessimistic view of the human condition?

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Friday May 1
Reading Quiz: *Hamlet*, Acts 3-5

1. Identify the speaker: "And for your part, Ophelia, I do wish / That your good beauties be the happy cause / Of Hamlet's wildness. So shall I hope your virtues / Will bring him to his wonted way again, / To both your honors."
   A. Polonius
   B. Queen Gertrude
   C. Laertes
   D. King Claudius

2. What does Hamlet make known to Ophelia in the first scene of Act 3?
   A. that he was the cause of his father's death
   B. that she was the cause of his father's death
   C. that he once hated her but now loves her
   D. that he once loved her but no longer does

3. How does Claudius respond to what he sees of the play *The Murder of Gonzago*?
   A. he expresses remorse over his grave offense
   B. he shows pride in the fact that he was not caught in his misdeeds
   C. he gives a prize to the players for their skillful acting
   D. he banishes the players from the realm for their poor acting

4. Who does Hamlet strike dead before knowing his identity?
   A. Claudius
   B. Polonius
   C. Laertes
   D. one of the players

5. Identify the speaker: "A man may fish with the worm that hath eat of a king and eat of the fish that hath fed of that worm."
   A. Polonius
   B. Claudius
   C. Rosencrantz
   D. Hamlet

6. To where does Claudius seek to send Hamlet?
   A. England
   B. Norway
   C. Poland
   D. France

7. Whom does the “rabble” threaten to make king?
   A. Hamlet
   B. Horatio
   C. Laertes
   D. Polonius

8. Who captures Hamlet and sends him back to Denmark?
   A. Fortinbras
   B. an unknown pirate
   C. English soldiers
   D. Polacks

9. What plan is hatched to ensure Hamlet is killed by “accident”?
   A. put him to sea in a ship that isn’t seaworthy
   B. poison him in his sleep
   C. place a candle in an unsafe place near his quarters
   D. convince him to agree to a friendly bout of fencing with Laertes

10. Whose grave is being prepared at the beginning of Act 5?
   A. Ophelia’s
   B. Polonius’s
   C. Hamlet’s
   D. Horatio’s

11. Identify the speaker: “I am justly killed with mine own treachery.”
   A. Claudius
   B. Osric
   C. Hamlet
   D. Laertes

12. True or false: Hamlet and Laertes make amends in their final moments.
   A. True
   B. False

13. In two to three sentences, answer the following question: What exactly is Hamlet asking when he poses the question, “To be or not to be” (3.1.64)?

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Remote Learning Packet

April 27-May 1, 2020
Course: 11 Precalculus
Teacher(s): Mr. Simmons

Weekly Plan:

Monday, May 4
- [ ] Story time!
- [ ] Read "Right triangles"
- [ ] Problems 1-5 odd

Tuesday, May 5
- [ ] Read "The Sine, Cosine, and Tangent functions"

Wednesday, May 6
- [ ] Problems 1-5 (all)

Thursday, May 7
- [ ] Read "Relationships between Trig Functions"

Friday, May 8
- [ ] Problems 1-8 (all)

Monday, May 4

1. Story time! If technologically feasible, send me an email telling me a fun story that happened recently. I miss you! (yes, you)

Learning from packets is much harder than learning in class. I've found (after much searching) a good resource for learning trigonometry. We'll be using it for the remainder of the year. It's the second part of a book called *Precalculus* by Mr. Joseph Gerth. I've included it as a material on Google Classroom.

2. Read "Right triangles" from Gerth's *Precalculus* (pp. 84-90).
3. Complete problems 1, 3, and 5 (pp. 90-91)

Tuesday, May 5

1. Read "The Sine, Cosine, and Tangent functions" (pp. 93-109).

Wednesday, May 6

1. Complete problems 1-5 (pp. 109-110) on the same piece of paper from Monday.

Thursday, May 7

1. Read "Relationships between the Trig functions" (pp. 113-121)

Friday, May 8

1. Complete problems 1-8 (pp. 121-122) on the same piece of paper from Monday and Wednesday.

Part two: Trigonometry

Trigonometry is the study of angles and their relationship to triangles. As you'll soon see, we attempt to relate the angles of a triangle to its sides. It is perhaps unsurprising that this Part is closely related to Geometry – we'll ask that famous question, "What is the relationship?" What is interesting, however, is that this Part is also closely related to Algebra. We will often work with equations and use algebra to change them into something different.

This reveals one reason why Trigonometry is studied: It is, in some sense, a union of those two great disciplines of Geometry and Algebra. It takes what you know in each discipline and applies it, so that you can expand each.

This branch of mathematics is also studied at this point because it is a very important Calculus concept. Calculus teachers and textbooks will assume that you can evaluate, e.g., $\sin\frac{\pi}{4}$ in seconds. You will also be expected to have deep insight into the graphs of the Trigonometric functions. Since you'll often be finding the area of a Trigonometric function or, perhaps, determining its behavior as it goes to infinity.

Many students struggle with Trigonometry. This is mostly likely due to their deficiencies in Geometry, Algebra, or both. We will give a small review of right triangle Geometry, which we think will be helpful for you. But many of you may need to do some reading, research, and practice on your own to get you up to par. Even when you are well-versed in the pre-requisites, you must also be prepared to spend time practicing and mastering new techniques as well.

Another issue that you must be aware of is that Trigonometry is very vertical. That is, anything learned serves as a foundation for the next thing. If you don't learn something, therefore, it's nearly impossible to progress. The cornerstone of this Part is most assuredly right triangles. You simply must master the special right triangles presented in the first section. Failure to do so will nearly guarantee failure. As such, you must approach each section with due diligence, and master any and all of the content.

Unit four

Right triangle Trigonometry

Τριγωνομετρία – Greek for Trigonometry. Literally meaning "triangle measuring."

Trigonometry is a branch of mathematics that looks at triangles, angles, and circles. It's a complex and often rigorous art that requires intuition over skills. It is imperative to develop and master the foundational knowledge of Trigonometry, because without it, success is very difficult.

Although we assume the reader knows nothing of Trigonometry, it might be wise to have some experience before beginning this Unit. Additionally, there are a few Geometry theorems that are assumed to be known. If any of the following seems difficult, it is highly recommended to consult the previous texts.

We begin by studying some basic Geometry. After this, we then introduce the basic Trigonometric functions: Sine, cosine, and tangent.

§1 Right triangles

Since the word "Trigonometry" literally means "triangle measure", perhaps it is no surprise that we begin working with triangles. Recall that a **triangle** is a **polygon** with three sides, and that a right triangle is so called because one of its angles measures 90° (which we define as a **right angle**). Thus, Figure 21 is a right triangle.

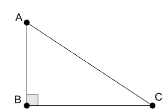


Figure 21

All right angles will have a box (like the one next to B) to denote the fact that they are 90°. Recall that it is not good enough to "look like" it's 90° - it *must* have the box.

This simple object contains many profound truths, some of which you may have learned in Geometry class. Perhaps the most well-known and profound truth is known as Pythagoras' Theorem. It has been known and used for millennia, contains more than 300 different proofs, and is used in just about every branch of mathematics.[i]

> Pythagoras' Theorem
>
> Given a right triangle with lengths a, b, c, where c is the **hypotenuse** and a, b are the **legs**, then
> $$a^2 + b^2 = c^2.$$

[i] Even statistics has a use for Pythagoras' Theorem!

We will usually prove each basic and foundational assertion in this course. However, we believe that most will have already proved this theorem on their own. Therefore we will not prove this assertion here.[ii]

Example 1

Find the length of the missing side in Figure 22.

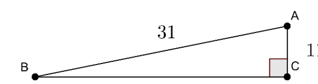


Figure 22

Pictures are not (and never will be) drawn perfectly to scale.

The hypotenuse – always opposite of the right angle – is side length $\overline{AB}$, which has a length of 31. Thus we can say that c, which represents the length of the hypotenuse in Pythagoras' Theorem, is 31. And since $\overline{AC}$ is one of our legs, we will let

$$a = 11.$$

Given this, we now have

$$11^2 + b^2 = 31^2$$

$$121 + b^2 = 961.$$

This is now a simple algebra problem. Solving for b we get

$$b^2 = 840$$

$$b = 28.98.$$

Some right triangles are special; they come in knowable ratios. And if we know the ratios of all three sides, then we do not need to use Pythagoras' Theorem at all.

Consider the right triangle shown in Figure 23.

[ii] The internet is a great resource. Use it.

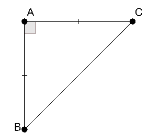
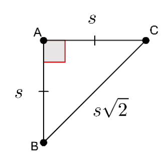


Figure 23

The tick marks on the two legs indicate that those two lines are the same length.

This is an **isosceles triangle**, which means that two of the side lengths are **congruent**. Consequently, we can make the following assertion.

Isosceles right triangle side ratio



Given an isosceles right triangle with a leg of length s, then the other leg has a length of s and the hypotenuse has a length of $s\sqrt{2}$.

Proof.

Construct isosceles right triangle ABC with leg $AB = s$. Then $\overline{AC} = s$ because we have an isosceles right triangle.

Further, we know that the length of the hypotenuse is

$$\overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2$$

by Pythagoras' Theorem. But we know the length of both $\overline{AB}$ and $\overline{AC}$, and so, by substitution, we have

$$s^2 + s^2 = \overline{BC}^2.$$

Whence, after simplification, we have

$$2s^2 = \overline{BC}^2$$

$$\overline{BC} = s\sqrt{2},$$

Which is what we wanted to show.

Example 2

Determine the lengths of the missing sides in Figure 24.

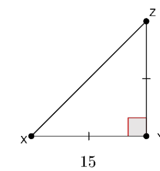


Figure 24

Applying the special triangle ratio, we have that $s = 15$. Thus the other leg,

$$\overline{YZ} = 15,$$

and the hypotenuse,

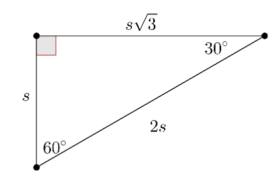
$$\overline{XZ} = 15\sqrt{2}.$$

Keep in mind Pythagoras's Theorem can still be used to determine the previous triangle, although you would have to apply the fact that $\overline{XY} = \overline{YZ}$.

Let's take a look at one more special right triangle ratio.

$30° - 60° - 90°$ Right triangle ratio

Given a right triangle with angles of $30°, 60°$, and $90°$, then the short leg has length s, the longer leg has a length of $s\sqrt{3}$, and the hypotenuse has a length of $2s$.



We will prove the previous result, but our proof will be a bit messy. We'll look for a better way to prove it as we continue through this Unit.

Proof.

To fully prove this assertion, we will need to prove three different possibilities:

I. A right triangle with legs of s and $s\sqrt{3}$,
II. A right triangle with a leg of s and a hypotenuse of $2s$, and
III. A right triangle with a leg of $s\sqrt{3}$ and a hypotenuse of $2s$.

We will prove the first point, then leave the final two proofs to the reader.

We are given a right triangle with lengths of s and $s\sqrt{3}$. Thus, using Pythagoras, we have

$$a = s, b = s\sqrt{3}$$

whence

$$s^2 + \left(s\sqrt{3}\right)^2 = c^2.$$

Consequently, we get that

$$s^2 + 3s^2 = c^2$$
$$4s^2 = c^2$$
$$c = 2s,$$

which is what we wanted to show.

Example 3

Complete the right triangle shown in Figure 25.

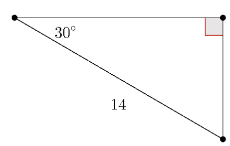


Figure 25

To "complete" a right triangle means to determine all of its side lengths and angles measures. In our case, we must determine one angle measure and the length of both legs. We begin with the angle measure. Recall from Geometry that the sum of three angles in a triangle is always 180°. Thus our missing angle (which we'll call α)[iii] is

$$\alpha = 180° - (30° - 90°)$$

[iii] It is my custom to name sides with lower-case Latin letters, a, b, c, and then call their opposite angles with lower-case Greek letters, α, β, γ, respectively. This is my own personal convention, and I welcome you to develop your own.

$$\alpha = 60°.$$

We'll use the ratios shown in our previous result to determine the side lengths. Recall from Geometry that that the shortest side of a triangle will always be opposite the shortest angle. Since, in a $30° - 60° - 90°$ triangle, the hypotenuse is always twice the length of the shortest side (which we'll call b), we can write

$$14 = 2b,$$

And hence

$$b = 7.$$

To determine the one remaining side, which is the longer leg (since it is opposite the 60° angle), we simply multiply our previous result by $\sqrt{3}$. Hence our remaining side (which we'll call a) is

$$a = 7\sqrt{3}.$$

We show our final results in Figure 26.

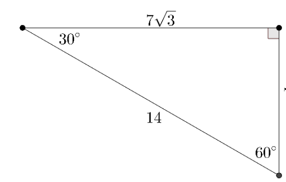


Figure 26

Pythagoras' Theorem allows us to determine the third side of a right triangle when only given two other sides. This is very useful, but it leaves out some other information. For example, what if we wanted to determine the angle measures when given two side lengths? Unless we measure it, we wouldn't be able to tell.[iv] Or, what if we had only known one of the sides of our triangle? Then we could not determine anything else.

This is one of the reasons that these special right triangles are so useful. We'll use that fact a myriad of times in this Part – and we'll come back to it very shortly. Doubtless, of course, you also see the advantage to solving a right triangle as quickly and efficiently as we just demonstrated.

[iv] And even if we did measure it, we would have to draw our triangle perfectly to scale, and even then, there is a degree of error in measurement.

Allow us to reiterate the importance of this section: You simply must become very adept at solving special right triangles. We cannot overemphasize how foundational this is. You will utilize this tool again and again in the forthcoming sections. We've given you many practice problems in this section, but you will need to take extra care to master it for yourself.

§1 Exercises

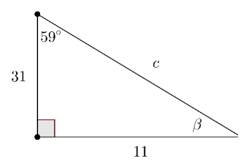
1.) In the following problems, assume that a and b are the lengths of legs, and that c is the length of a hypotenuse. Find the length of the missing side.
   - (A) $a = 12, b = 15$
   - (B) $a = 1.5, b = 2.3$
   - (C) $a = 24, b = 8$
   - (D) $a = 12.5, c = 19$
   - (E) $b = 113, c = 201$
   - (F) $c = 4.5, a = 1$
   - (G) $c = 27, b = 24$
   - (H) $a = 10, b = 1$

2.) Assume you have a triangle with leg lengths of 5 and 7.
   - (A) Troy lets $a = 5$ and $b = 7$. Tina lets $a = 7$ and $b = 5$. Do they get the same result for c?
   - (B) Prove that a and b are interchangeable when using Pythagoras' Theorem.

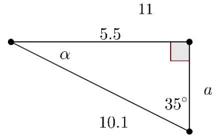
3.) Are all triangles possible? For example, can you have a right triangle with legs of length 10 and 20 and a hypotenuse of 15? Why or why not?

4.) Recall that all triangles' angles add up to 180°. With this in mind, complete the following right triangles.

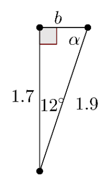
(A)



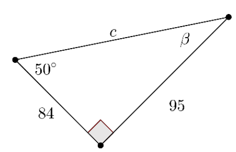
(B)



(C)



(D)



5.) So far, we've seen that $a^2 + b^2 = c^2$ is true for all real numbers – that is, at least one of the three lengths was irrational. Is it possible that all three lengths are integers? Let's explore…
   - (A) $a = 3, b = 4, c = ?$
   - (B) $a = ?, b = 12, c = 13$
   - (C) See if you can discover three other **Pythagorean Triples**.

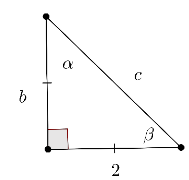
6.) The following table represents a small list of Pythagorean Triples. (Each row contains the three side lengths of the triangle.)

Side lengths	3	4	5
Side lengths	6	8	10
Side lengths	9	12	15

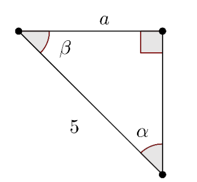
   - (A) What pattern exists in this table?
   - (B) Use the pattern to list out three more Pythagorean Triples.
   - (C) How many Pythagorean Triples exist?
   - (D) These were all multiples of the smallest (and most common) Pythagorean Triple, the 3,4,5 triangle.[v] Will this trick work with other distinct Pythagorean Triples, like the 5,12,13 triangle?

7.) Complete the following right triangles.

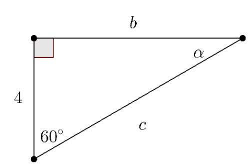
(A)



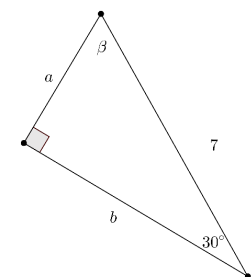
(B)



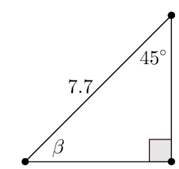
(C)



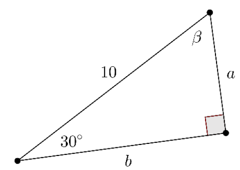
(F)

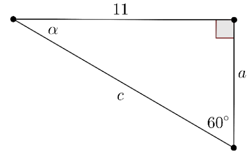


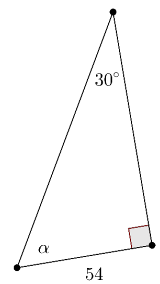
(G)



[v] We highly encourage you to memorize the 3,4,5 triangle. It pops up enough that memorizing it will save you some considerable time.

(D)


(E)


(H)


8.) As a matter of convention, we often rewrite any ratios that have an irrational number in their denominator such that the numerator is entirely rational. For example,

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

In most cases, this is purely by convention. One reason for doing this is that you will only need to memorize one set of numbers when it comes to specific Trig ratios.

(A) What is the denominator in the example shown above? And what is the numerator and denominator of the number multiplied by $\frac{1}{\sqrt{2}}$?

(B) What is $\frac{\sqrt{2}}{\sqrt{2}}$?

(C) Based on your answer to (B), has the value of $\frac{1}{\sqrt{2}}$ changed after the multiplication? Use a calculator if you're not convinced.

(D) Use the above example to rationalize the following denominators.

i. $\frac{3}{\sqrt{3}}$
ii. $\frac{4}{\sqrt{2}}$
iii. $\frac{\sqrt{2}}{\sqrt{3}}$
iv. $\frac{\pi}{\sqrt{7}}$
v. $\frac{3}{\sqrt{4}}$
vi. $\frac{5}{2\sqrt{2}}$
vii. $\frac{24}{\sqrt{12}}$
viii. $\frac{\sqrt{3}}{\sqrt{10}}$

9.) Answer True or False.

(A) A $45° - 45° - 90°$ triangle can never have three side lengths which are all whole numbers.

(B) The shortest side of a triangle is always opposite of the shortest angle.

(C) The hypotenuse is always the longest side of a triangle.

(D) A triangle can never have more than one 90° angle.

(E) There is an infinite amount of Pythagorean Triples.

10.) In your own words, describe the shortcut to completing an isosceles right triangle.

11.) In your own words, describe the shortcut to completing a $30° - 60° - 90°$ triangle.

12.) Will you forget these special triangle ratios?

13.) Where will you look if you forget them?

14.) You're sure you won't forget them??

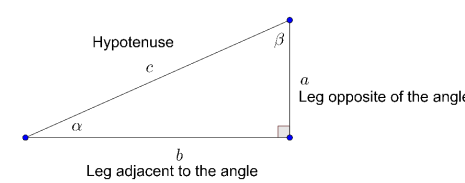
§2 The Sine, Cosine, and Tangent functions

We used Geometry in the previous section (and we will frequently fall back on it) and it had some good uses, but in this section we will discover a new set of tools that will allow us to do even more.

More importantly, though, is that we'll be able to relate angle measures and side lengths. Certainly, this will allow us to calculate more than we could previously, but we will also find a shred of truth that we felt was there but couldn't quite ascertain.

To begin with, we'll define a function that takes an acute angle as its input and returns a ratio of two side lengths.

The Sine function



The Sine function accepts as an input an acute angle of a right triangle, and then returns the ratio of the side opposite to the angle to the hypotenuse. Symbolically,

$$\sin \alpha = \frac{a}{c}.$$

We could also state that

$$\sin \beta = \frac{b}{c}.$$

Note that our current definition only accepts acute angles of a right triangle.

Example 1a

Write the ratio of the sine of angle α and β given Figure 27.

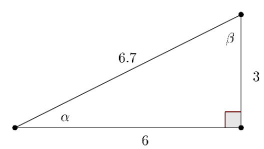


Figure 27

Let's start with the sine of angle α, which is denoted as $\sin\alpha$. This is just a ratio – so all we need to do is create a fraction. Since we've defined this function as the side opposite to the hypotenuse, we just need to write

$$\sin\alpha = \frac{3}{6.7},$$

Since 3 is the side opposite and 6.7 is the hypotenuse. It really is that simple!

The sine of angle β is handled the same way, except notice that the side opposite β is different than the side opposite of α. In this case we have

$$\sin\beta = \frac{6}{6.7}.$$

Whatever angle α is, when we input it into the sine function, we receive as an output $\frac{3}{6.7}$. The same is true for β; when we input β into the sine function, we receive as an output $\frac{6}{6.7}$.

Of course, that hasn't really revealed to us any new information. Let's try a problem that teaches us something, shall we?

Example 1b

Evaluate $\sin 30°$ given Figure 28.

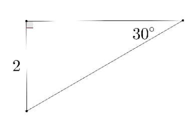


Figure 28

To answer this, we simply need to set up our ratio. Based on what we've been given, we have

$$\sin 30° = \frac{2}{\text{hypotenuse}}.$$

But what is the length of the hypotenuse? We can easily find that out using what we learned in the previous section.[i] Since 2 is the length of the short leg, then the hypotenuse must be twice that length. Hence we conclude that they hypotenuse has a length of 4. Therefore

$$\sin 30° = \frac{2}{4} = \frac{1}{2}.$$

So what did we find out? Well, if you input 30° into the sine function, you should get out $\frac{1}{2}$. But there's a very important question to answer: Will this only happen when the short leg of a $30° - 60° - 90°$ triangle has a length of 2? Put another way, does the size of the triangle play a role in determining the result of $\sin 30°$?

To answer this question, we will do as all mathematicians do: We will explore, we will experiment, we will play! We'll make a new triangle with an angle of 30° that has a different size and observe what happens.

Example 1c

Evaluate $\sin 30°$ using Figure 29.

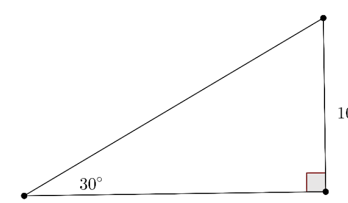


Figure 29

[i] This is exactly what we mean we say that you will need to be very adept at the previous section. This is often how we'll calculate our Trig functions – by setting up a special right triangle, and filling in the rest of the information. Therefore, we'll state it again: Learn the previous section!

We apply the same procedure. In this case, the hypotenuse must be 20. Since the sine ratio is $\frac{\text{opposite}}{\text{hypotenuse}}$, and the side length opposite to 30° is 10, we conclude that

$$\sin 30^\circ = \frac{10}{20} = \frac{1}{2}.$$

Now wait just a minute, that's very interesting! Why did a different size triangle give us the same result? This is most unexpected!

Indeed, much of mathematics was discovered in the same way. It starts with a simple curiosity, and then a conscious effort to poke and prod whatever you have until it gives up some profound truth.[ii] Think of math like a piñata, which contains so much delicious truth. But the only way to get at it is to swing at it a few times (and oftentimes this swinging is truly difficult!).

In this case, the profound truth seems to be that $\sin 30^\circ = \frac{1}{2}$ *no matter what size triangle we have*. Thus we share the following very important theorem.

The dependency of $\sin\alpha$
The output of $\sin\alpha$ only depends on α. The size of the triangle is irrelevant.

There is, of course, a reason that $\sin\alpha$ does not depend on the size of the triangle, and we'll briefly explore this in the Exercises.

This profound truth will allow us to construct *any* size triangle we wish when we go about determining the sine of any angle. This is useful, since some choices will be easier than others. In fact, it is this very fact that will motivate our work in the next unit. Let's briefly explore this idea.

Example 1d

Evaluate $\sin 45^\circ$.

Based on what we just learned, we should create a $45^\circ - 45^\circ - 90^\circ$ triangle, and then create a ratio using the lengths of the opposite side and the hypotenuse. It would be fair to ask what triangle, exactly, you should create. We start with the basics in Figure 30.

[ii] And this process of experimentation and verification can be as quick and painless as we just witnessed, or it could take several years. Or centuries. But it's the journey that counts, right?

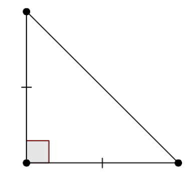


Figure 30

We've drawn some isosceles right triangle. Now how should we label the angles and the side lengths? Certainly, the two non-right angles must be 45° (since they are both equal and must equal 90°), but what about the side lengths? We could, for example, start with the legs, and label each, say, 10. Once we've made that decision, we must conclude that the hypotenuse is $10\sqrt{2}$.[iii] We draw this in Figure 31.

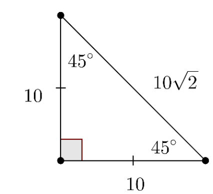


Figure 31

There's nothing *wrong* with the previous picture, but it also isn't the *best*. Should we choose a small number? Or, shouldn't we choose some number where we won't have any simplification?

Because of this, why don't we let the legs be of length 1.[iv] Then we have Figure 32.

[iii] See the previous section.
[iv] You may not see that this is the "simplest" number. That's fine! This is something you find out through experimentation and exploration.

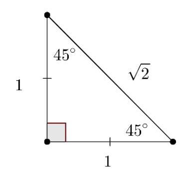


Figure 32

To calculate $\sin 45°$, then, we just need to set up our ratio accordingly.[v] We have

$$\sin 45° = \frac{1}{\sqrt{2}}.$$

What if we had chosen the original triangle, where the legs had a length of 10? Then we would have had

$$\sin 45° = \frac{10}{10\sqrt{2}} = \frac{1}{\sqrt{2}},$$

Which is the same answer, although it was one less step to get to the answer. In future sections, we'll be picking a very particular triangle to work with when we want to evaluate some Trig function. We are just being clever, and trying to save ourselves some time – although this is not mandatory, and you are free to pick any triangle you like.[vi]

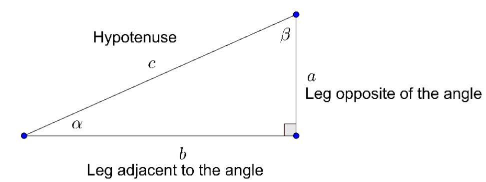
A single step may seem insignificant, but since we'll be going back to these two special triangles again and again,[vii] you'll want to streamline the process as much as possible. Regardless, we now have another value or ratio to add to our list, which you'll work on completing in the Exercises.

One more note before we continue on: Most textbooks will tell you that $\sin 45° = \frac{\sqrt{2}}{2}$. Are they wrong? And where did they get that answer from? And why would they bother writing it like that? Good questions!

Let us now move on to the Cosine function. By its very name, we can infer that it is related to the sine function, and – as we'll soon see – this is very much the case.

[v] Will it matter which 45° angle we choose? See for yourself!
[vi] We will reiterate this point in future sections.
[vii] And again!

The Cosine function



The Cosine function accepts as an input an acute angle of a right triangle, and then returns the ratio of the side adjacent to the angle to the hypotenuse. Symbolically,

$$\cos\alpha = \frac{b}{c}.$$

We could also state that

$$\cos\beta = \frac{a}{c}.$$

The Cosine function is very similar to the Sine function. It, too, accepts an angle as an input. But this time it returns as an output the side length adjacent[viii] to α over the hypotenuse. This – again – has only been defined for acute angles in a right triangle.

Example 2a

Write the Cosine ratio of α given Figure 33.

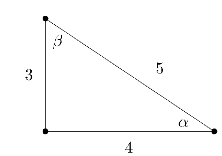


Figure 33

All we need to do is make a fraction of the length of the adjacent leg over the hypotenuse. This is simple: We get

[viii] Note that adjacent means touching. And while, yes, the hypotenuse is touching α, what we really mean is the adjacent leg.

$$\cos\alpha = \frac{4}{5}.$$

It is important to note that these ratios will *only* work with right triangles. Was the above triangle a right triangle? It might *look* that way, but how can we be sure?[ix] Up to this point, we've always seen the box at the right angle. Here we don't have one – so is our result incorrect?

This is an excellent question and one you must be aware of. Pythagoras' Theorem tells us that for any right triangle,

$$a^2 + b^2 = c^2.$$

The converse of Pythagoras' Theorem – that is, if $a^2 + b^2 = c^2$ then you have a right triangle – is also true. Since

$$3^2 + 4^2 = 5^2,$$

we can conclude we have a right triangle. Thus the aforementioned conclusion (that $\cos\alpha = \frac{4}{5}$) is correct.

Example 2b

List out the Sine and Cosine ratios for angles α and β in Figure 34.

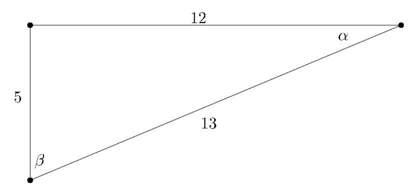


Figure 34

Remember to first verify that you have a right triangle. Since

$$5^2 + 12^2 = 13^2,$$

we can move on to the next step.

[ix] For example, if the unnamed angle, call it γ, is equal to 89.99 (which is really close to 90) then our Trig ratios are inaccurate. We must be positive we have an actual right triangle; it is not good enough to *almost* be a right triangle.

Let us deal with α first. Our ratios for α are

$$\sin\alpha = \frac{5}{13}$$

and

$$\cos\alpha = \frac{12}{13}.$$

Then, with respect to β, we have

$$\sin\beta = \frac{12}{13}$$

and

$$\cos\beta = \frac{5}{13}.$$

But isn't there something peculiar about our results? Look again – is there anything that you can see?

Why is it that

$$\sin\alpha = \frac{5}{13} = \cos\beta\,?$$

Or that

$$\sin\beta = \frac{12}{13} = \cos\alpha\,?$$

This is a note-worthy find! Why should two different functions (with different inputs) produce the same output?

Before we formalize our next theorem, let us consider the relationship that exists between the two non-right angles in a right triangle.

First, let us (in Figure 35) draw some right triangle.

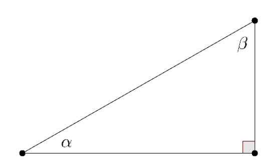


Figure 35

Let us label the two non-right angles as α and β. Is there some relationship between them?

We now need some sort of starting point – so let's think back to other relationships in triangles. The biggest one, perhaps, is the theorem that the sum of all three angles in a triangle are 180°. So let's begin with this, play with it a bit, and then see if we can't uncover something of note.[x]

Using the previous relationship, we would then have

$$\alpha + \beta + \gamma = 180.$$

But we know $\gamma = 90°$, since it is a right angle (and by definition, all right angles equal 90°). Substituting this in, we see that

$$\alpha + \beta = 90°.$$

This yields us one important relationship: α and β are complementary. This seems benign… But let us consider the word *Cosine* for a moment. It is actually a shortened form of the term "Complement of the sine." Wait a moment… Is it a coincidence that we just showed that α and β are complementary and the Cosine function has in its very name the word "complement"? Most certainly not![xi] We've discovered some profound truth! We now provide a formal proof.

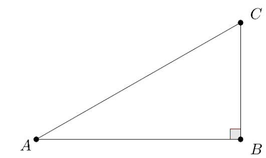
Proof.

We want to show that the two non-right angles of a right triangle must be complementary, that is, that their sum must be 90°.

Construct right ΔABC with right angle B.

[x] Of course, our exploration may turn out fruitless. That's OK! Believe it or not, finding out what doesn't work can also very helpful. Additionally, don't get discouraged if you're not correct the first time – very few mathematicians are!

[xi] Which doesn't mean "Never". It just means unlikely. Some may scoff at the simple logic we used here, but keep in mind we're showing a process of coming to an informal conclusion. You may wish the explanation was more formal, and if that's the case, I commend you and encourage you to write your own textbook. ☺



Because the sum of all three angles in a triangle is 180°, we know that $A + B + C = 180°$. But $B = 90°$, since (by definition) all right angles are equal to 90°. Then, by substitution, we have that $A + 90° + C = 180°$. Using the subtraction property of equality reveals that $A + C = 90°$, which is precisely what the word "complementary" means.

We have thus shown what we wanted to show.

With this in mind, let us formalize the previous.

The relationship between the Sine and Cosine functions. The two non-right angles in a right triangle are always complementary.

Example 2c

Evaluate $\cos 30°$.

We go through the same procedure found in Examples 1c and 1d. We first draw a right triangle[xii] (Figure 36).

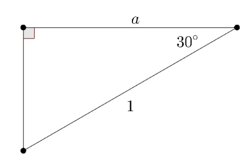


Figure 36

We chose to make our hypotenuse 1 here. There are a few other nice choices, as well.

Based on the triangle that we chose, we have that

$$\cos \alpha = \frac{\text{adjacent}}{1}.$$

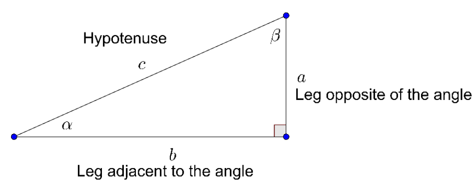
[xii] Remember that we are free to draw any right triangle we want. Well, as long as is it has a 30° angle, anyway.

But what is the length of the leg adjacent to α? We can figure this out because this is a special right triangle, viz., it is a $30° - 60° - 90°$ triangle. Since our hypotenuse is 1, the short leg (not marked) must measure $\frac{1}{2}$. Then, to find the long leg, we multiply $\frac{1}{2}$ by $\sqrt{3}$. Thus the long leg, a, must be $\frac{\sqrt{3}}{2}$. Since $a = \frac{\sqrt{3}}{2}$, we conclude that

$$\cos\alpha = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}.$$

We have one more Trigonometric function to uncover. We'll define it here, but in the exercises you will *derive*[xiii] it yourself.

The Tangent function



The Tangent function accepts as an input an acute angle, and then returns the ratio of the side opposite to the angle to the side adjacent to the angle. Symbolically,

$$\tan\alpha = \frac{a}{b}.$$

We could also state that

$$\tan\beta = \frac{b}{a}.$$

The Tangent function appears a bit different from the Sine and Cosine function, and, to be sure, there are some key differences. But you'll also discover in the exercises that it is closely related to the Sine and Cosine functions! Additionally – as you'll soon see – we work with it in the same manner.

Example 3a

Write the Sine, Cosine, and Tangent ratios using α of the triangle shown in Figure 37.

xiii What this means is that you'll take something you already know, work with it, and come out with the function we're about to define! Obviously this is the better way to do it, but we don't want to spoil your fun.

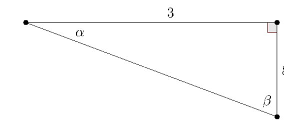


Figure 37

All we need to do is substitute. The only problem we have is that we don't have a length for the hypotenuse. No matter – this is easily found. We use Pythagoras' Theorem and find that the hypotenuse must have a length of 8.54. With this in mind, we have

$$\sin\alpha = \frac{8}{8.54},$$

$$\cos\alpha = \frac{3}{8.4},$$

and

$$\tan\alpha = \frac{8}{3}.$$

Example 3b

Evaluate $\tan 45°$.

We follow the same procedure as Examples 2c, 1c, and 1d. We create any size right triangle with a 45° angle, as in Figure 38.

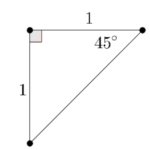


Figure 38

Recall that any $45° - 45° - 90°$ triangle will also be isosceles. Both legs, therefore, are also congruent.

Now all we need to do is write our ratio. We have

$$\tan 45° = \frac{1}{1} = 1.$$

Thus we conclude that $\tan 45°$ is 1.

Usually, evaluating a Trig function given any input is very difficult to do. Of course, in today's world, we have ready access to calculators, which are very good at the tedious process of approximating, so it isn't too difficult to evaluate $\sin 2°$, for example. You'll need to have a calculator on hand.

Example 4a

Determine the length of a given Figure 39.

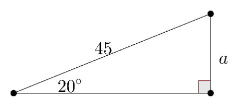


Figure 39

As we did extensively in Geometry, we should first establish some relationship. For example, we know (from Pythagoras) that $a^2 + b^2 = c^2$. In this case, however, we don't know a or b, and so Pythagoras' Theorem yields us no help. Since we have been given an angle measure and one side length, it stands to reason that we should use a Trig function. We have three different choices (Sine, Cosine, and Tangent), so which one should we use? In this example, we will try all of them and show you that there is really only one good choice.[xiv] First, let us try the Sine function. Given that $\sin \alpha = \frac{\text{opposite}}{\text{hypotenuse}}$, we have (after substitution)

$$\sin 20° = \frac{a}{45}.$$

If we try the Cosine function, we have

$$\cos 20° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{?}{45},$$

where the question mark represents the unknown length of the adjacent.

If we try the Tangent function, we would have

$$\tan 20° = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{?},$$

where, again, the question mark represents the same unknown length of the adjacent.

[xiv] This won't always be the case. Sometimes there will be two or three good choices.

Based on what we just saw, the Sine is the function we will choose. This is because we have an equation,

$$\sin 20° = \frac{a}{45},$$

which can be solved for the missing variable a. If we used the Cosine function, we could find the length of the adjacent, but that would not get us the length of a.[xv] The Tangent function is even worse; we would have two variables in our equation, which would put us in an untenable position.

To solve this equation, we should evaluate the left side, and then get the variable all by itself. Using a calculator, we find that $\sin 20° \approx 0.342$. Hence

$$0.342 = \frac{a}{45}$$

$$15.39 \approx a.$$

The key to these problems is to identify the correct relationship. Then it's just a matter of substituting and solving, both of which are basic Algebra skills.

Example 4b

Determine the length of b given Figure 40.

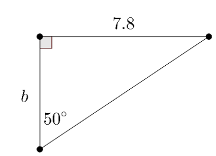


Figure 40

We face a similar problem. Here the best relationship to use is Tangent, since we can then form the equation

$$\tan 50° = \frac{7.8}{b}.$$

Then we just solve the equation whence

[xv] That being said, we could find the length of a after we find the length of the adjacent. It's not an ideal process though, as we have to go through the entire procedure again, or, alternatively, use Pythagoras' Theorem. Either way, it's much more efficient to use the Sine function.

$$b \approx 6.54.$$

Example 4c

Complete the right triangle shown in Figure 41.

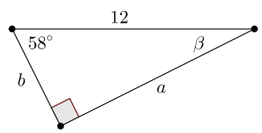


Figure 41

Recall that when we are asked to "complete" a right triangle, it just means that we should find out the length of every side and the measure of every angle.[xvi]

We have some freedom in how we approach this problem, but there are also some limitations.[xvii] Let us first find a. To do this, we must ask that ever-popular question: What is the relationship? In this case, we could[xviii] use the Sine function to get

$$\sin 58° = \frac{a}{12}$$

$$0.85 \approx \frac{a}{12}$$

$$a \approx 10.18.$$

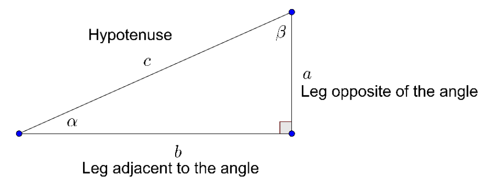
Now that we have a, we can easily find b by using Pythagoras' Theorem. We can also find β easily, since the sum of the three angles must be 180°. Hence

$$b \approx 6.35, \beta = 32°.$$

Before you begin practicing these concepts, there are three other Trig functions which exist. These are rarely used by themselves, but they will be of some importance later. We mention them here for the sake of completion.

[xvi] That makes a total of six things we're looking for, although we must be given at least three of them to determine the other three.
[xvii] For example, could we find any of the sides using Pythagoras' Theorem?
[xviii] We say "could" because we could also find the length of b by using the Cosine function. Could we use the Tangent function here? Why or why not?

The reciprocal Trigonometric functions



The Cosecant function: $\csc \alpha = \frac{c}{a}$

The Secant function: $\sec \alpha = \frac{c}{b}$

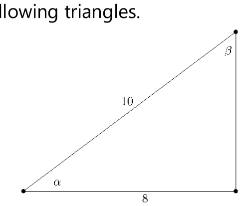
The Cotangent function: $\cot \alpha = \frac{b}{a}$

The name "reciprocal Trig functions" is very apropos. See if you can find out why.

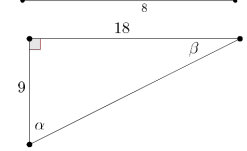
§2 Exercises

1.) Write the ratios of the Sine, Cosine, and Tangent function (for both α and β) given the following triangles.

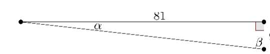
(A)



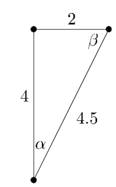
(B)



(C)



(D)



2.) Our next goal is to create a table for the Sine, Cosine, and Tangent functions. First, however, we'll only focus on those really nice angles which produce special right triangles.

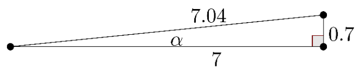
(A) What are the "really nice angles" of which we speak? (Hint: Think of the angles seen in all of our special right triangles.)

(B) Draw any two triangles with the information from (A). Make sure you label the sides and angles.

(C) Now create a table of values for the Sine, Cosine, and Tangent functions. Use (and expand) the incomplete table below as a model.

α	$\sin\alpha$
30°	$\frac{1}{2}$
45°	$\frac{\sqrt{2}}{2}$

3.) We have not yet defined $\sin 0°$ and $\cos 0°$, so let's do that now. Consider the triangle below, where α is a very small angle.



Note that, in the above case, $\sin\alpha$ will be relatively small (which you should verify).

(A) What will happen to the length of the side opposite of α as the angle of α decreases?

(B) Finally, imagine that $\alpha = 0$. What will the length of the side opposite to α be?

(C) Given your answer for (B), what must $\sin 0°$ be equal to?

(D) Now let's consider the Cosine function. Already, the length of the side adjacent to α and the length of the hypotenuse are very close. What is $\cos\alpha$ given the picture?

(E) As α gets close to zero, what will happen to the length of the hypotenuse in relation to the length of the adjacent side? (Hint: Try drawing smaller and smaller angles for α)

(F) Given your answer for (E), what must $\cos 0°$ be equal to?

(G) Verify your results for (C) and (F) with a calculator.

4.) Now let's define the results of $\sin 90°$ and $\cos 90°$.

(A) We should try to use a similar approach as the previous problem. Draw a right triangle where α is close to 90°, but not quite that large.

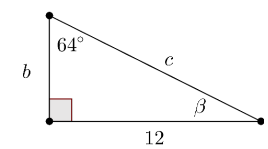
(B) Based on your picture, what will $\sin 90°$ and $\cos 90°$ be?

(C) Verify your results with a calculator.

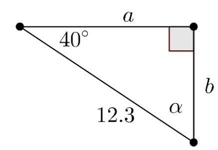
5.) Create a table of values for all three Trig functions from 0° to 90°. Increment each row by 5° (so your first number is 0°, then 5°, then 10°...). You should use a calculator for this problem.

6.) Complete the following right triangles.

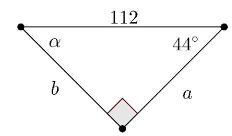
(A)



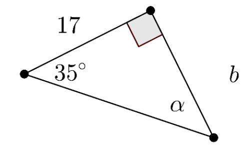
(B)



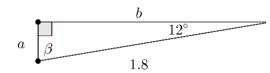
(C)



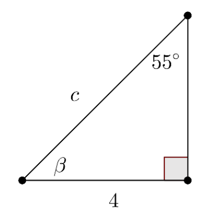
(D)



(E)



(F)



7.) The previous concepts can also be applied to quasi-real scenarios.

(A) Benny leans a ladder against a 24 foot wall, so that the top of the ladder touches the top of the wall perfectly. The angle that is created with the ground and the ladder is 70°. Determine the length of the ladder.

(B) Building A is shorter than Building B. To determine how much shorter, the mayor lays a tape measure from Building A to Building B and sees that the slant height is 45 m. Then he also figures out that the angle between the tape measure and Building B is 62°. How much shorter is Building A than Building B?

(C) A water slide is housed inside a tower that is 50 ft tall. The water slide travels in a straight line down toward the ground, where it hits the ground at a 24° angle. How much space is needed on the ground for this slide?

(D) A tree's shadow is 10 ft long. The angle that the sun creates with the flat ground is 80°. Determine the height of the tree.

8.) Create your own quasi-real situation, similar to one of the previous problems, that requires Trigonometry to solve.

9.) Look at the table of values you created for the Sine function.

(A) What is the maximum value of $\sin\alpha$?

(B) What is the minimum value of $\sin\alpha$?
(C) Are those maximum and minimum values true for any value of α? Try $\sin 500°$ or $\sin(-100°)$ to verify.
(D) Describe any patterns you see with the Sine function.

10.) Look at the table of values you created for the Cosine function.
(A) What is the maximum value of $\cos\alpha$?
(B) What is the minimum value of $\cos\alpha$?
(C) Are those maximum and minimum values true for any value of α?
(D) Describe any patterns you see with the Cosine function.

11.) Look at the table of values you created for the Tangent function.
(A) What is the maximum value of $\tan\alpha$?
(B) What is the minimum value of $\tan\alpha$?
(C) Are those maximum and minimum values true for any value of α?
(D) Describe any patterns you see with the Tangent function.

12.) Imagine for a moment that you measure a triangle in yards, and you find that the two legs measure 3 and 4 yards, while the hypotenuse measures 5 yards. You send this information to your friend who lives in a European country that normally uses meters instead.
(A) Calculate $\sin\alpha$ using the initial correct measurement that used yards.
(B) Now convert yards into meters (round to the nearest hundredth). Calculate $\sin\alpha$ in meters.
(C) Assume that when you sent this triangle to your friend, you did not label your measurements. So your friend assumes you mean 3 meters, 4 meters, and so on. Calculate $\sin\alpha$ with this incorrect information.
(D) Was your answer any different in (A), (B), or (C)? What does that tell you about Sine and Cosine ratios (viz., do units matter)?

13.) One popular mnemonic device for remembering the different Trig ratios is SOH CAH TOA. Explain what this means and how it helps you to remember how to set up your ratios. (If you don't know it, look it up)

14.) Let us now derive the Tangent function. To accomplish this, we only need the Sine and Cosine function.
(A) When deriving things, it is often useful to rewrite them in their most basic terms. Before we can do that, however, let us draw a right triangle. Choose any non-right angle and call it α. Then label the sides (according to your choice of α) as "opposite", "adjacent", and "hypotenuse".
(B) Using your picture, write out the Sine and Cosine ratios.
(C) Is there any way we can take our Sine and Cosine ratio and turn them into our Tangent ratio? Recall that Tangent is $\frac{\text{opposite}}{\text{adjacent}}$. Try adding, subtracting, multiplying, or dividing them.
(D) Why is it better to derive the tangent function than to define it? Put another way, what issues would you run into if you just defined everything in math?

§3 Relationships between the Trig functions

In our final section, we'll look at how the three main Trig functions are related to one another. We already know a few relationships, viz. that the Cosine function is the complement to the sine function. So it is fair to believe that there are other relationships that exist, and in this section, we will tease them out. We will spend our time deriving and formalizing many of the basic and fundamental **identities** in this section. An identity is a different way to write an equivalent statement. For example, $2 + 2 = 4$ is an identity; $2 + 2$ is simply a different name for 4.

Before we embark on learning some identities, let us first look at a more basic question: Does knowing one Trig ratio lead us to find any of the others? Put another way, if we know one Trig ratio, can we find *all* of the other?

Example 1a

If $\sin\alpha = \frac{3}{5}$, write the other six Trig ratios.

Let us first draw a picture. Because the Sine ratio is the side opposite to α to the hypotenuse, it makes sense to draw Figure 42.

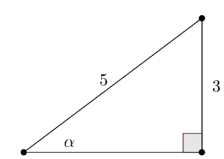


Figure 42

As before, this isn't the *only* triangle that you could draw.

This drawing makes our life much easier. Now what we'll do is determine the other ratios. Starting with Cosine, we would have

$$\cos\alpha = \frac{\text{adjacent}}{5}.$$

But what is the length of the adjacent side? This can be easily figured out with Pythagoras' Theorem, although, in this case, we recognize that we have a Pythagorean Triple, and thus the adjacent side is 4. With this in mind, the rest of the Trig ratios are elementary. They are

$$\cos\alpha = \frac{4}{5}, \qquad \tan\alpha = \frac{3}{4}, \qquad \csc\alpha = \frac{5}{3}, \qquad \sec\alpha = \frac{5}{4}, \qquad \cot\alpha = \frac{4}{3}.$$

Example 1b

If $\tan\beta = \frac{5}{7}$, write the other six Trig ratios.

You should, again, start with a picture.[i] We draw the triangle shown in Figure 43.

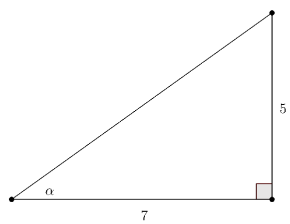


Figure 43

Before we write out any of our ratios, how about we find the length of the missing side, which in the present case is the hypotenuse. Since we do not recognize this as a Pythagorean Triple, we go ahead and use Pythagoras' Theorem. We find that

$$5^2 + 7^2 = c^2$$

$$c = \sqrt{74}.$$[ii]

With this value, we can now write all six ratios:

$$\sin\alpha = \frac{5}{\sqrt{74}}, \qquad \cos\alpha = \frac{7}{\sqrt{74}}, \qquad \tan\alpha = \frac{5}{7}, \qquad \csc\alpha = \frac{\sqrt{74}}{5}, \qquad \sec\alpha = \frac{\sqrt{74}}{7},$$
$$\cot\alpha = \frac{7}{5}.$$

[i] Sensing a pattern yet?
[ii] You should always leave this number in exact form. Simplify it if you can.

Note that it isn't necessary to rationalize the denominators, although it might be good practice for you to do so. Also note that some textbooks and standardized tests will require you to do this, so make sure you know how to do this.

Example 1c

Given that $\sin\alpha = a$, find all six Trig ratios.

This seems more difficult than the previous problems. Do not let first appearances intimidate you – you just need to draw a picture. The only issue with our picture is that we only seem to have the length of one side, a. But recall that $a = \frac{a}{1}$, so we can draw our picture as seen in Figure 44.

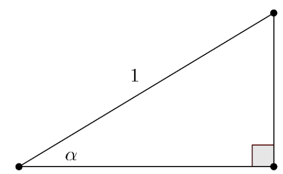


Figure 44

So what's the length of the adjacent side? We can figure this out using Pythagoras' Theorem, which, unsurprisingly, is exactly what we did in the previous Examples. We let b be the length of the adjacent side, and set up our equation like so:

$$a^2 + b^2 = 1^2.$$

We will let you finish this Example in the Exercises.

We have seen how we can relate one Trig ratio to the others. We will next relate the functions themselves.

A good starting point is to begin with what we know, and build from there. We know that

(1) $$\cos\alpha = \sin\beta \text{ iff } \alpha + \beta = 90°.$$

This is a fine relationship, but let's make this more useful. One issue is that there are two variables, so let's try to simplify that statement.

Example 2a

Rewrite (1) in terms of α.[iii]

We are given that $\cos\alpha = \sin\beta$ iff $\alpha + \beta = 90°$. We can thus make a substitution. Since

$$\alpha + \beta = 90°$$

then we know that

$$\beta = 90° - \alpha$$

and hence

$$\cos\alpha = \sin(90° - \alpha).$$

Example 2b

Given $\cos 20°$, what is the equivalent cofunction?

To answer this, we need only use the previous result. In the present case, $\alpha = 20°$ and hence

$$\cos 20° = \sin(90° - 20°) = \sin 70°.$$

The answer we're looking for is $\sin 70°$. A calculator can quickly verify that indeed, $\cos 20° = \sin 70°$.

Do recognize that the cofunction of Cosine is Sine, and vice versa. Which Trig function is the cofunction of Tangent? Or how about Cosecant? Is there an easy way to tell?

Example 2c

Write all of the **cofunctions identities**.

We will not write all of them, but instead, will write one of them and leave the rest to the reader.

$$\sin\alpha = \cos(90° - \alpha).$$

Other cofunction identities exist, and you will have to write them out in your exercises.

The cofunction identities can be useful, but mostly they just highlight the relationship between the Sine and Cosine function (and the other co-Trig functions).

[iii] In other words, there should only be one variable, α.

Let us now explore what happens when we square the Sine or Cosine functions. First of all, if we write $\sin\alpha^2$ there could be a bit of confusion.[iv] So, if we mean $(\sin\alpha)(\sin\alpha)$, we will write

$$\sin^2\alpha.$$

If we mean $\sin(\alpha^2)$, where the angle α is squared but not the function, we'll write it as $\sin(\alpha^2)$.

Example 3

Evaluate $\sin^2\alpha$ given Figure 45.

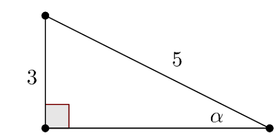


Figure 45

Recall that $\sin^2\alpha = (\sin\alpha)(\sin\alpha)$, and since $\sin\alpha = \frac{3}{5}$, we have, by substitution,

$$\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)$$

$$\frac{9}{25}.$$

You'll work to explore squared Trig functions in the exercises.

Now let's see what happens when we add two squared functions.

Let's try $\sin^2 45° + \cos^2 45°$. Since $\sin 45° = \frac{\sqrt{2}}{2}$, we can conclude that

$$\sin^2 45° = \frac{2}{4} = \frac{1}{2}.$$

This is also true of $\cos^2 45°$, since $\cos 45° = \frac{\sqrt{2}}{2}$. Thus we have

$$\sin^2 45° + \cos^2 45° = \frac{1}{2} + \frac{1}{2} = 1.$$

This is quite nice. Will this always be true?

[iv] Are we squaring the angle α or squaring the function $\sin\alpha$?

Example 4

Evaluate $\sin^2 60° + \cos^2 60°$.

From the previous section, we have memorized that $\sin 60° = \frac{\sqrt{3}}{2}$.[v] Thus $\sin^2 60° = \frac{3}{4}$. We also have memorized $\cos 60° = \frac{1}{2}$, and thus $\cos^2 60° = \frac{1}{4}$. We thusly conclude that

$$\sin^2 60° + \cos^2 60° = \frac{3}{4} + \frac{1}{4} = 1.$$

But that's strange… Why have we gotten the same answer? And why is that answer so pleasant? Have we stumbled upon some remarkable truth?

> Pythagorean identity
>
> For any angle α,
> $$\sin^2 \alpha + \cos^2 \alpha = 1.$$

We'll revisit this identity in graphical format, but that will be after we reveal the Unit Circle. It is there that we will prove this identity, as it is very easy to do when we view it graphically.

It would be safe to ask if this is the only such relationship between cofunctions. For example, is $\tan^2 \alpha + \cot^2 \alpha$ also equal to one? You will explore that in the exercises.

In §2, we briefly introduced the Reciprocal Trig functions, such as Secant. These are defined as the reciprocals of their respective Trig function. In case, for example, we want to find the ratio of the hypotenuse to the opposite, we would then use the Cosecant function. And since

$$\csc \alpha = \frac{\text{hypotenuse}}{\text{opposite}}$$

while

$$\sin \alpha = \frac{\text{opposite}}{\text{hypotenuse}},$$

we recognize that these are reciprocals (thus the name).

Recall that one way we can write a reciprocal is to put it under one. Let's put that into symbols. If α is some number, and we want to find its reciprocal, we can just evaluate (or

[v] And if you haven't memorized it, that's ok too. You will then need to construct a $30° - 60° - 90°$ triangle and write out the ratio, then simplify what you have.

leave, if we prefer) $\frac{1}{\alpha}$. So if we want to find the reciprocal of the number $\frac{3}{4}$, we just need to evaluate $\frac{1}{\frac{3}{4}}$, which is $\frac{4}{3}$, as you should verify.

With this in mind we can now define the Reciprocal identities.

> Reciprocal identities
> $$\sin \alpha = \frac{1}{\csc \alpha}, \qquad \csc \alpha = \frac{1}{\sin \alpha}$$

Of course, there are other reciprocal identities, but you will write the remainder in your exercises.

Let us now put these identities to use.

Example 5a

Simplify $\frac{\sin \alpha}{\csc \alpha} + \frac{\cos \alpha}{\sec \alpha}$.

The key to this problem is to rewrite this expression into something easier. So what we'll want to do is identify some aspect of the expression that can be rewritten. In this case, let's try rewriting $\csc \alpha$ and see what we get. Since

$$\frac{\sin \alpha}{\csc \alpha} = \sin \alpha \cdot \frac{1}{\csc \alpha}$$

and $\frac{1}{\csc \alpha} = \sin \alpha$, we can say that

$$\frac{\sin \alpha}{\csc \alpha} = \sin^2 \alpha.$$

This same procedure will reveal that

$$\frac{\cos \alpha}{\sec \alpha} = \cos^2 \alpha.$$

So we can rewrite each addend in our original problem and we end up with

$$\sin^2 \alpha + \cos^2 \alpha,$$

which isn't too bad. But we can rewrite that expression as something even more simple! Since that is a Pythagorean identity and is equal to one, our final result is simply

$$1.$$

How did we know to change $\csc \alpha$ and $\sec \alpha$? Nothing but intuition – in other words, when we work with these sorts of problems, there is no prescribed method to simplify the expression. Experience is a big help, but do not underestimate planning and patience, either. And of course, you must be well-versed in the various identities that we've learned so far. You'll find that if you don't know the identities very well that simplifying expressions in this manner will be very difficult. No matter how good you become at these identities, however, be prepared to spend some time with them. Even very good mathematicians sometimes struggle with these, so don't feel incompetent if you don't get the answer right away.

One rule of thumb which we'll reiterate to you: You will most likely want to convert any Trig functions into Sine or Cosine, if possible. We have many identities which work with Sine and Cosine, but only a few that work with the reciprocal Trig functions. This isn't always the case, but it's usually the best place to start.

Example 5b

Simplify $\frac{\sin\alpha}{\cos\alpha}$.

You should have already found this out in the Exercises in the previous section, but this one is so important that going over it a second time will be helpful to you. Note that the procedure we use here will very rarely be used by you in your Exercises. But again, the result is so important we feel it necessary to include.

One way we can rewrite $\sin\alpha$ and $\cos\alpha$ is in terms of their ratios. But ratios require a triangle, right? And we don't have one, so what shall we do? Well, how about we make one? Consider Figure 46, which will allow us to find the ratios of our two Trig functions.

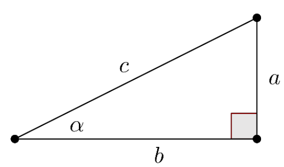


Figure 46

This isn't the only way you can draw this triangle, either. The key is just to draw one.

With this in mind, we can now substitute, since $\sin\alpha = \frac{a}{c}$ and $\cos\alpha = \frac{b}{c}$, we have

$$\frac{\sin\alpha}{\cos\alpha} = \frac{\frac{a}{c}}{\frac{b}{c}}.$$

Recall that one can divide two fractions by multiplying by the reciprocal. Thus we have

$$\frac{a}{c}\cdot\frac{c}{b}.$$

But $\frac{c}{c}$ is one, and therefore the c values cancel out. This leaves us with

$$\frac{a}{b}.$$

This is a ratio! And in fact, it is no more than the Tangent function's ratio. Thus we conclude that

$$\frac{\sin\alpha}{\cos\alpha} = \tan\alpha.$$

You will use this identity many, many times, and so we formalize it below.

Quotient identity

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$$

Example 5c

Simplify $\cos\alpha\cdot\tan\alpha$.

Following our rule of thumb, let's convert everything into Sines and Cosines. To do this, we simply rename $\tan\alpha$ using our identity above. We then have

$$\cos\alpha\cdot\frac{\sin\alpha}{\cos\alpha}$$

whence we see that the Cosines cancel. This simply leaves us with

$$\sin\alpha.$$

This process of simplifying expressions will be formalized in Unit six. Until then, the goal is to introduce you to this kind of thinking.

§3 **Exercises**

1.) Use the given Trig ratio to write out all six Trig ratios in each problem.

(A) $\sin\alpha = \frac{12}{13}$
(B) $\tan\alpha = \frac{4}{3}$
(C) $\cos\alpha = \frac{7}{25}$
(D) $\sin\alpha = \frac{1}{4}$
(E) $\cos\alpha = \frac{10}{13}$
(F) $\tan\alpha = \frac{5}{6}$
(G) $\csc\alpha = \frac{7}{2}$
(H) $\sec\alpha = \frac{10}{7}$

2.) Let us complete Example 1c. If $\sin\alpha = a$, what are the six Trig ratios?

3.) Now let $\tan\alpha = d$.
   (A) What are the six Trig ratios given this?
   (B) Compare this answer to the previous.

4.) In Example 1c, we let the length of the hypotenuse of the triangle to equal 1. Is that OK? Why don't we let the hypotenuse equal c, to allow for any and all possibilities? Now suppose $\sin\alpha = \frac{x}{c}$, and the lengths of your triangle are a, b, and c, with the hypotenuse equaling c.
   (A) What are the six Trig ratios?
   (B) Compare this with the previous two results.

5.) Given the Trig functions and angle measure, write the equivalent cofunction.
   (A) $\sin 30°$
   (B) $\cos 10°$
   (C) $\cot 7°$
   (D) $\sec 64°$
   (E) $\cos 31°$
   (F) $\tan 14°$
   (G) $\csc 47°$
   (H) $\sin 25°$
   (I) $\tan(\beta+\gamma)$
   (J) $\sin\beta$

6.) Write out all of the cofunction identities, including the ones we discovered in the reading. Hint: There are six of them.

7.) Now write out all of the reciprocal identities. Hint: There are six of them.

8.) Evaluate the following.
   (A) $\sin^2 30°$
   (B) $\cos^2 30°$
   (C) $\tan^2 30°$
   (D) $\cos^2 45°$
   (E) $\tan^2 45°$
   (F) $\sin^2 60°$
   (G) $\cos^2 60°$
   (H) $\tan^2 60°$
   (I) $\sin(30°^2)$
   (J) $\cos^2\alpha$

9.) Write out a table of values for $\sin^2\alpha$, $\cos^2\alpha$, and $\tan^2\alpha$, starting at $\alpha = 0$, going up by 5° each row, and ending at $\alpha = 90°$. You will need a calculator for this Exercise.

10.) Use your results from the previous Exercise to answer the following questions.
   (A) What is the maximum value of $\sin^2\alpha$, $\cos^2\alpha$, and $\tan^2\alpha$?
   (B) What is the minimum value of $\sin^2\alpha$, $\cos^2\alpha$, and $\tan^2\alpha$?
   (C) Are there any similarities or differences between $\sin^2\alpha$ and $\sin\alpha$? Compare your results from the previous section.

11.) One of the most important relationships in Trigonometry is the Pythagorean Identity we discussed in the reading. Write this identity down now.

12.) Evaluate the following.
   (A) $\sin^2 60° + \cos^2 60°$
   (B) $\cos^2 30° + \sin^2 30°$
   (C) $\sin^2\left(\frac{\alpha}{2}\right) + \cos^2\left(\frac{\alpha}{2}\right)$
   (D) $\sin^2(3\alpha+\pi) + \cos^2(3\alpha+\pi)$

13.) It is often helpful to rewrite $\sin^2\alpha$ or $\cos^2\alpha$. Use the Pythagorean Identity to rewrite $\sin^2\alpha$ and $\cos^2\alpha$.

14.) Simplify the following.
   (A) $\tan\alpha\cdot\csc\alpha$
   (B) $(\sin\alpha+\cos\alpha)^2$

15.) Are there any other Pythagorean Identities? To find this out, use a calculator and try the following for different values of α.
   (A) $\sec^2\alpha + \csc^2\alpha$
   (B) $\tan^2\alpha + \cot^2\alpha$
   (C) There are two other Pythagorean Identities. First, using the previous two, guess what they might be. Then, if you can't figure it out, look them up and write them down now. We'll discover how to arrive at these results when we have some better tools.

16.) Answer True or False.
   (A) $\sin\alpha = \cos(\alpha - 90°)$
   (B) $\sin^2\alpha + \cos^2\beta = 1$ iff $\alpha+\beta = 90°$
   (C) $\sin^2\alpha = \sin\alpha\cdot\alpha$
   (D) $\sin^2\alpha$ is sometimes negative.[vi]

[vi] Assume $\alpha \in \mathbb{R}$.

GreatHearts
Irving

Remote Learning Packet

NB: Please keep all work produced this week. Details regarding how to turn in this work will be forthcoming.

April 27 - May 1, 2020
Course: Spanish III
Teacher(s): Ms. Barrera anna.barrera@greatheartsirving.org
Supplemental links: www.conjuguemos.com www.spanishdict.com

Weekly Plan:

Monday, April 27
☐ Capítulo 5 - Read an announcement for a job position then choose the appropriate answer..
☐ Capítulo 5 - Translate your answers.

Tuesday, April 28
☐ Capítulo 5 - Read the curriculum of Ms. Diaz and then choose the appropriate answer..
☐ Capítulo 5 - Translate your answers.

Wednesday, April 29
☐ Capítulo 5 - Read the curriculum of Mr. Perez and then choose the appropriate answer..
☐ Capítulo 5 - Translate your answers.

Thursday, April 30
☐ Capítulo 5 - Read about the history of Spain from the Celtiberos to the conquest of the muslims by Fernando e Isabela.
☐ Capítulo 5 - Second, match and complete with the appropriate vocabulary according to your comprehension of the reading.

Friday, May 1
☐ Capítulo 5 - Vocabulary Quiz A and B
☐ Capítulo 5 - Vocabulary Quiz - C

Statement of Academic Honesty

I affirm that the work completed from the packet is mine and that I completed it independently.

I affirm that, to the best of my knowledge, my child completed this work independently

Student Signature

Parent Signature

Monday, April 27

Capítulo 5 - Read an announcement for a job position and then choose the appropriate answer. Then translate your answers for each number. **A google document has been added in google classroom** to type out your answers and submit within the lesson of the day. Please be organized and number your answers.

I. Handout: Trabajo y comunidad: Lectura 1 - **Read the Lectura a**nd then choose the appropriate answer.
II. Handout: **Lectura 1 - Translate your answers.**

Tuesday, April 28

Capítulo 5 - Read Ms. Diaz's Curriculum and then choose the appropriate answer. Then translate your answers for each number. **A google document has been added in google classroom** to type out your answers and submit within the lesson of the day. Please be organized and number your answers.

I. Handout: Curriculum de Karen Díaz Manzanares: Lectura 2 - **Read the Lectura a**nd then choose the appropriate answer.
II. Handout: **Lectura 2 - Translate your answers.**

Wednesday, April 29

Capítulo 5 - Read the curriculum of Mr. Perez and then choose the appropriate answer. Then translate your answers for each number. **A google document has been added in google classroom** to type out your answers and submit within the lesson of the day. Please be organized and number your answers.

I. Handout: Curriculum de Brandon Perez: Lectura 3 - **Read the Lectura a**nd then choose the appropriate answer.
II. Handout: **Lectura 3 - Translate your answers.**
III. Writing: Integration of ideas. Compare both curriculums. Then, according to your observations, then decide which of the two candidates is best prepared for the job from Lectura 1. Use information from the announcement and both curriculums to support your ideas.

Thursday, April 30

Capítulo 5 - Read about the history of Spain from the Celtiberos to the conquest of the muslims by Fernando e Isabela. Second, match and complete with the appropriate vocabulary according to your comprehension of the reading. **A google document has been added in google classroom** to type out your answers and submit within the lesson of the day. Please when typing your answers label Exercise A, B, etc.

I. **Spanish Reading** with Comprehension Exercises. *La historia De España*. **Exercise A**. Choose the appropriate letter. **Exercise B**. Choose the appropriate phrase in order to complete the sentence. **Exercise C.** Select the sentence that is false(falso) and rewrite it making it true. (cierto). For example, 1.

F then rewrite the sentence to make it true. **Exercise D**. Fill in the blank with the appropriate word from the reading.

Friday, May 1

Capítulo 5 - Comprehension of work and volunteer vocabulary.

I. Quiz 5-1: Comprensión del vocabulario 1. Type in your answers in the google doc provided for you in google classroom. A. Type in your 5 answers in the google doc. B. Type in your 8 answers in the google doc for this section.
II. Quiz 5-1:   Comprensión del vocabulario 1. Type in your answers in the google doc provided for you in google classroom for C. Because there are no numbers, please add them with your answers. There are 6 for the top and 6 for the bottom.

Nombre ____________________ Fecha ____________________

Tema 5 Trabajo y comunidad: Lectura 1

Anuncio clasificado

MAESTRO/A DE ESPAÑOL

Colegio Colombo Bilingüe necesita un/a maestro/a de español a tiempo completo para un puesto de secundaria. El candidato ideal también debe tener un interés en el tenis y en participar en equipos deportivos. Debe tener buenas destrezas de comunicación y disponibilidad para trabajar los fines de semana para las actividades deportivas.

Requisitos indispensables:

- Experiencia mínima de 4 años
- Título universitario completo
- Bilingüe/nivel avanzado de español
- Conocimientos de computación
- Tres cartas de recomendación

Ofrecemos muy buen salario y buenos beneficios. Para completar la solicitud de empleo, aquellos interesados favor de enviar su carta de interés y una copia de su currículum por correo electrónico o fax al (979) 456-9087.

Nombre ______ Hora ______

Tema 5 Fecha ______

Vocabulario y comprensión

1. **Vocabulario** Según el contexto del clasificado, ¿quién tiene un "título universitario completo"?
   - **A** una persona que entra en la universidad
   - **B** una persona que estudia en la universidad
   - **C** una persona que se graduó de la universidad
   - **D** una persona que quiere ir a la universidad

2. **Vocabulario** Según el texto, ¿cuál de estas opciones **NO** es un requisito indispensable?
   - **A** tener diploma de la secundaria
   - **B** tres cartas de recomendación
   - **C** más de 5 años de experiencia
   - **D** conocimientos de computación

3. **Vocabulario** El texto dice que el/la candidato/a debe tener "disponibilidad para trabajar los fines de semana". Según la descripción, ¿qué se espera que haga el candidato los fines de semana?
   - **A** preparar las clases de español
   - **B** revisar la tarea
   - **C** trabajar en actividades extracurriculares
   - **D** usar sus conocimientos de computación

4. **Composición y estructura** Según el clasificado, ¿qué se puede inferir sobre el candidato ideal para el puesto de maestro/a?
   - **A** Es una persona atlética.
   - **B** No necesita saber usar tecnología.
   - **C** Trabajó de maestro menos de cuatro años.
   - **D** No es muy deportista.

Nombre ______ Hora ______

Tema 5 Fecha ______

Vocabulario y comprensión (continuación)

5. **Ideas clave y detalles** ¿Quién sería un candidato ideal para esta posición?

A una maestra de español de primaria con 10 años de experiencia

B un joven de Colombia con experiencia en la enseñanza del tenis

C una joven graduada de España que sabe computación

D una profesora de español con mucha experiencia y que practica varios deportes regularmente

Nombre ______ Fecha ______

Tema 5 Trabajo y comunidad: Lectura 2

Currículum de Karen Díaz Manzanares

Datos personales

Nombre y apellidos: KAREN DÍAZ MANZANARES

Teléfono: (620) 913 8460

Lugar y fecha de nacimiento: Lima, Perú, 8 de abril de 1982

Correo electrónico: Karen.díazman@gmail.com

Formación académica

Máster de Enseñanza (*Teaching*) en Educación Secundaria, Facultad de Pedagogía, Universidad Autónoma de Lima 2009–2010

Licenciatura en Literatura, Universidad Nacional de Autónoma de Lima 2001–2005

Experiencia profesional docente

Colegio Sagrado Corazón octubre 2007 – presente

Funciones: Maestra de Español e Historia en cursos de ESO y Bachillerato

Colegio Virgen del Remedio febrero 2005 – junio 2007

Funciones: Monitora y coordinadora de actividades extracurriculares

Experiencia profesional no docente

Hotel Parque Central 2005–2006

Funciones: Reservas, atención al cliente, información turística

Guía-Arte SL 2003–2004

Funciones: Relaciones públicas, realización de rutas culturales y gastronómicas

Cursos y seminarios

El uso de nuevas tecnologías en la clase, 100 horas (UCLA). Año 2010

Técnicas de ayuda a estudiantes con dificultades en el español, 110 horas (Instituto Cervantes). Año 2010

Los Materiales Didácticos en el Aula, 35 horas (Instituto Cervantes). Año 2009

Idiomas extranjeros

Inglés: Nivel alto. *Certificate of English* (Boston University Extension)

Francés: Nivel medio.

Nombre ______ Hora ______

Tema 5

Fecha ______

Vocabulario y comprensión

1. **Vocabulario** Vuelve a leer la sección "Formación académica". Según el texto, ¿cuál es un sinónimo de la palabra "formación" en este contexto?

   A educación

   B formalidad

   C pedagogía

   D funciones

2. **Vocabulario** Vuelve a leer las secciones "Experiencia profesional docente" y "Experiencia profesional no docente". ¿Con qué se relaciona la palabra "docente" en el contexto de la lectura?

   A hacer tareas en la oficina de una escuela

   B practicar deportes

   C dar clases y hacer otras actividades escolares

   D trabajar en hoteles y asistir a clientes

3. **Vocabulario** Contesta las preguntas.

   Parte A: Vuelve a leer la sección "Formación académica" y observa la palabra "licenciatura". ¿Qué palabra de esa sección **NO** te ayuda a entender su significado?

   A formación académica

   B máster

   C universidad

   D literatura

   Parte B: Identifica la definición apropiada de la palabra "licenciatura".

   A estudios de secundaria

   B estudios de primaria

   C estudios universitarios

   D estudios de posgrado

Nombre ______ Hora ______

Tema 5

Fecha ______

Vocabulario y comprensión (continuación)

4. **Composición y estructura** ¿Cuál es el propósito de este tipo de texto (currículum)?

   A informar sobre la experiencia de vida y trabajo de una persona

   B contar una autobiografía en forma cronológica

   C informar sobre las aficiones y los pasatiempos de una persona

   D contar la vida y obra de una persona

5. **Composición y estructura** Observa cómo está organizado y estructurado este texto en diferentes secciones y puntos. ¿Cuál es el propósito de organizar el currículum de esta forma?

   A presentar información con muchos detalles

   B presentar información en forma objetiva

   C presentar información personal en forma de narración

   D presentar información personal en un modo breve y claro

6. **Ideas clave y detalles** Identifica qué tipo de trabajo le puede interesar más a Karen, según sus estudios y su experiencia de trabajo más reciente.

   A enseñar y trabajar en escuelas

   B trabajar en restaurantes

   C trabajar en agencias de viaje

   D dar cursos y seminarios

7. **Ideas clave y detalles** Según la información que presenta Karen, ¿qué puedes inferir acerca de cuáles pueden ser algunos de sus pasatiempos? Escoge **dos** opciones.

   A mirar televisión y salir de compras

   B viajar y probar nuevos restaurantes

   C practicar deportes

   D aprender nuevos idiomas

Nombre ______________________ Fecha ______________

Tema 5 **Trabajo y comunidad: Lectura 3**

Currículum de Brandon Perez

Brandon Perez
593 Rockaway Valley Road
Denver, CO 88012
teléfono: 720-919-0846
correo electrónico: branpe@gmail.com

OBJETIVO

Encontrar un puesto de maestro de español para educación primaria o secundaria

EXPERIENCIA DOCENTE

Eanes Middle School, Denver, CO — agosto 2011–presente

Maestro de Español de séptimo y octavo grado

- Desarrollar y diseñar actividades interactivas y virtuales para las clases de español
- Entrenar al equipo de tenis en el otoño y al equipo de fútbol en la primavera
- Colaborar en la organización del torneo de atletismo
- Coordinar la Feria de Comida Internacional de Estudiantes

Coast Redwood Bilingual School, Felton, California — agosto 2008–2011

Maestro de secundaria

- Enseñar Ciencias Sociales, Español, tenis, baloncesto y fútbol a estudiantes de séptimo y octavo grado
- Organizar excursiones escolares
- Organizar el torneo anual de vóleibol interescolar

EXPERIENCIA NO DOCENTE

Redwood Tenis Club, Felton, California — agosto 2008–2011

Instructor de tenis

- Dar clases de tenis a niños entre 6 y 16 años de nivel principiante, intermedio y avanzado
- Organizar torneo de fin de año
- Responsable de mantener el equipo

California Pizza Kitchen, Santa Cruz, California — septiembre 2004–junio 2005

- Preparar pizzas

Nombre ______________________ Fecha ______________

Tema 5

Trabajo y comunidad: Lectura 3 (continuación)

EDUCACIÓN

Máster de Educación, University of California Los Ángeles	junio 2007
Bachelor of Arts, de Inglés y Español, University of California Santa Cruz	junio 2005
• Estudios en el extranjero: Salamanca, España	2003–2004

IDIOMAS

Español: Nivel avanzado

Italiano: Nivel medio

DESTREZAS DE COMPUTACIÓN

Habilidad en los siguientes programas: Atlas Rubicon, Canvas, Promethean Interactive White Board, iWork, Microsoft Office. Experiencia con sistemas PC y Mac.

INTERESES

Profesionales: Desarrollo profesional y pedagógico, integración de tecnología en la clase

Personales: Tenis, ciclismo, yoga, tomar fotografías, escalar y viajar

Nombre ____________________ Hora ________

Tema 5

Fecha ____________________

Vocabulario y comprensión

1. **Vocabulario** Vuelve a leer las secciones "Experiencia docente" y "Experiencia no docente". ¿Cuál de las siguientes expresiones usa el autor como un sinónimo de la palabra "enseñar"?

   A dar clases

   B diseñar actividades

   C organizar torneos

   D entrenar equipo

2. **Vocabulario** Vuelve a leer la sección "Destrezas de computación". ¿Cuál de estas palabras te ayuda a entender el significado de la palabra "destrezas"?

   A experiencia

   B sistema

   C computación

   D habilidad

3. **Ideas clave y detalles** Según la información personal, profesional y los estudios de este candidato, ¿qué se podría inferir acerca del origen de este candidato y su español?

   A Es de España y el español es su primer idioma.

   B Es estadounidense y el español es su segundo idioma.

   C Es de España y es completamente bilingüe en inglés y español.

   D Es de los Estados Unidos y no habla mucho español.

4. **Ideas clave y detalles** Según la experiencia no docente y los intereses de Brandon, ¿cuáles pueden ser algunos de sus pasatiempos?

   A Le gusta quedarse en casa los fines de semana.

   B Le gusta escalar y ver películas.

   C Le gusta practicar deportes.

   D Le gusta cocinar y salir de compras.

Nombre _______________ Hora _______________

Tema 5

Fecha _______________

Vocabulario y comprensión (continuación)

5. **Ideas clave y detalles** Según la experiencia profesional de Brandon, ¿qué otros deportes además del tenis podría entrenar?

   A fútbol y baloncesto

   B baloncesto y vóleibol

   C atletismo y fútbol

   D vóleibol y ciclismo

6. **Integración de conocimientos** Los currículum de las lecturas 2 y 3 muestran diferencias culturales entre los currículum del mundo hispano y los de los Estados Unidos. Según el orden de las secciones en cada uno, ¿qué se puede inferir acerca de la importancia que Brandon le da a la experiencia de trabajo a diferencia de Karen?

   A La experiencia es más importante que sus datos personales.

   B La experiencia vale más que los idiomas adicionales que habla.

   C La experiencia es más importante que la educación.

   D La experiencia es tan importante como la educación.

7. **Integración de conocimientos** Identifica otra diferencia cultural entre los currículum del mundo hispano y los de los Estados Unidos, según los modelos de las lecturas 2 y 3.

   A Tienen algunas diferencias pero no son importantes ya que contienen la misma información.

   B En los países de habla hispana, los currículum no incluyen los cursos adicionales.

   C En los países de habla hispana, los currículum suelen incluir la foto del candidato/a.

   D En los Estados Unidos, los currículum no hacen referencia a los intereses personales.

Chapter 36
La historia de España

HABITANTES PRIMITIVOS

1. Los **iberos** y los **celtas** fueron los primeros habitantes de España. De la unión de iberos y celtas se formó la raza celtíbera, precursor de los españoles de hoy.
2. Los **fenicios** fueron una nación de marineros. Fundaron la ciudad de Cádiz. Los griegos fundaron colonias en la costa oriental de España. Estos dos grupos estaban en España del siglo once al siglo ocho, antes de Cristo, aproximadamente.
3. Los **cartagineses** invadieron a España en el siglo III antes de Cristo. Una nación guerrera, vencieron a los celtíberos en la batalla de Sagunto.
4. Los **romanos** vencieron a los cartagineses alrededor del año 200 antes de Cristo y reinaron en España durante seis siglos (hasta el año 400 después de Cristo, aproximadamente). Dejaron mucha influencia en España: dieron a España la religión cristiana, las leyes romanas, y su lengua (el castellano se deriva del latín). También construyeron puentes, acueductos y caminos.
5. Los **visigodos,** una tribu germánica, vencieron a los romanos e invadieron a España por el año 409 después de Cristo. Reinaron hasta la invasión de los moros.
6. Los **moros** invadieron a España y vencieron a los visigodos en 711. Se quedaron en España por siete siglos, hasta 1492, y dominaron el país durante gran parte del tiempo. Dejaron mucha influencia en la vida española. Esta influencia morisca incluye la introducción de la noria para regar los campos, la introducción de las ciencias (matemáticas, medicina, astronomía, filosofía, etc.), la construcción de palacios, alcázares y mezquitas, y la introducción de nuevas palabras en el idioma español, sobre todo palabras que comienzan con *al-* (algodón, alcalde, álgebra, etc.).

RECONQUISTA DE ESPAÑA

1. **Pelayo** venció a los moros en la batalla de Covadonga en 718.
2. **El Cid** (Rodrigo Díaz de Vivar), el héroe nacional de España, capturó la ciudad de Valencia de los moros en 1094.
3. En 1492, después de más de siete siglos de lucha, los españoles, bajo **Fernando** e **Isabel** (los Reyes Católicos), vencieron finalmente a los últimos moros, en Granada. Así terminó la Reconquista.

ESPAÑA, UN PAÍS PODEROSO

Los Reyes Católicos —Fernando, el rey de Aragón, e Isabel, la reina de Castilla—terminaron la Reconquista de España y empezaron su unificación. Bajo su reinado, España llegó a ser la primera nación de Europa. Ayudaron al navegante genovés Cristóbal Colón a realizar su empresa y le dieron tres carabelas, la Niña, la Pinta y la Santa María. Colón y sus marineros pisaron tierra el 12 de octubre en una isla que llamaron San Salvador. Con el descubrimiento del Nuevo Mundo empezó el gran imperio español.

Carlos V (1516-1556), nieto de Fernando e Isabel, fue uno de los reyes más poderosos de España. Durante su reinado, la época de los grandes descubrimientos y conquistas, España llegó a tener posesiones en Europa y en el Nuevo Mundo.

Felipe II fue hijo de Carlos V. Durante su reinado España tomó parte en muchas guerras. Venció a los turcos en la batalla de Lepanto (1571). Su «Armada Invencible» fue vencida por Inglaterra en 1588.

DECADENCIA DE ESPAÑA (SIGLO XVIII)

La decadencia de España se debió, entre muchas cosas, a las guerras, los altos impuestos y a los reyes débiles que gobernaron al país durante esa época.

LOS SIGLOS XIX Y XX

1. La **Guerra de Independencia** (1808-1814), comenzó el 2 de mayo de 1808 con una rebelión contra los franceses (Napoleón). Esta fecha se convirtió en la fiesta nacional del país. Lograron expulsar a las fuerzas francesas en 1814.
2. Las **Guerras Carlistas** (1833-39; 1872-76) fueron largas guerras civiles en las cuales Carlos, hermano de Fernando VII, trató de quitarle el trono del país a Isabel, hija de Fernando VII.
3. La **Guerra Hispanoamericana** entre España y los Estados Unidos tuvo lugar en 1898. España fue vencida y perdió sus posesiones ultramari-

nas de Cuba, Guam, Puerto Rico, y las Islas Filipinas.

4. La Primera República fue establecida en 1873, y duró solamente un año. Alfonso XII estableció otra vez la monarquía. Su hijo, Alfonso XIII, fue rey de España hasta 1931. Durante su reinado, estableció una dictadura bajo Primo de Rivera. La segunda república fue establecida el 14 de abril de 1931.
5. En 1936 estalló una guerra civil que duró hasta 1939. El general Francisco Franco venció a las fuerzas de la república y estableció una dictadura. Franco rigió (*ruled*) el país hasta su muerte en 1975. El gobierno actual es una monarquía constitucional. En 1975, Juan Carlos I fue proclamado rey por las Cortes. En 1982, Felipe González Márquez llegó a ser presidente del Gobierno. A partir de 1985, España es miembro de la Comunidad Económica Europea (C.E.E.).

EXERCISE A. **A la izquierda de cada expresión de la lista A, escriba la letra de la expresión correspondiente de la lista B.**

A	B
________ *1.* Pelayo	*a.* el Cid
________ *2.* los Reyes Católicos	*b.* Napoleón
________ *3.* Rodrigo Díaz de Vivar	*c.* rey de España hasta 1931
________ *4.* Felipe II	*d.* los Estados Unidos
________ *5.* Dos de Mayo	*e.* Fernando e Isabel
________ *6.* Alfonso XIII	*f.* Armada Invencible
________ *7.* guerra que ocurrió al fin del siglo XIX	*g.* monumento romano
________ *8.* puente de Alcántara	*h.* 14 de abril de 1931
________ *9.* la segunda república	*i.* dictador español
________ *10.* Francisco Franco	*j.* principió la Reconquista

EXERCISE B. **Escoja y subraye la palabra o expresión que complete correctamente cada frase.**

1. Los primeros habitantes de España fueron los (iberos, moros, romanos).

____

2. La segunda república española se estableció en el año (718, 1931, 1808).

____

3. El acueducto de Segovia fue construido por los (fenicios, romanos, moros).

____

4. Muchas palabras españolas que principian con "al-" son de origen (romano, árabe, ibero).

____

5. Las guerras entre los moros y los españoles se llamaron (las Guerras Civiles, las Guerras Carlistas, la Reconquista).

____

6. España perdió a Cuba, Puerto Rico, y las Filipinas en (la guerra civil, la guerra con los Estados Unidos, la guerra de independencia).

____

7. Pelayo venció a los moros en la batalla de (Covadonga, Lepanto, Sagunto).

____

8. Los visigodos vencieron a los (moros, romanos, fenicios).

____

9. El dictador de España bajo Alfonso XIII fue (Francisco Franco, Primo de Rivera, Pelayo).

____

10. El hijo de Carlos V fue (Fernando VII, Felipe II, Alfonso XIII).

____

EXERCISE C. **Indique si cada frase es cierta o falsa. Si es falsa, cámbiela para hacerla cierta.**

1. Los romanos reinaron hasta la invasión de los moros.

____

2. Se debe la introducción de la noria a los cartagineses.

____

3. Pelayo venció a los moros en la batalla de Sagunto.

____

4. Los españoles vencieron a los turcos en la batalla de Lepanto.

____

5. España llegó a tener muchas posesiones ultramarinas durante el reinado de Fernando e Isabel.

____

6. La «Armada Invencible» fue vencida por los ingleses en 1588.

____

7. La primera república española duró cinco años.

____

8. La decadencia de España se debió a muchas guerras, reyes débiles, y la expulsión de los judíos y los moros.

____

9. La Guerra Civil comenzó en 1936 y no terminó hasta 1941.

6. España perdió a Cuba, Puerto Rico, y las Filipinas en (la guerra civil, la guerra con los Estados Unidos, la guerra de independencia).

____

7. Pelayo venció a los moros en la batalla de (Covadonga, Lepanto, Sagunto).

____

8. Los visigodos vencieron a los (moros, romanos, fenicios).

____

9. El dictador de España bajo Alfonso XIII fue (Francisco Franco, Primo de Rivera, Pelayo).

____

10. El hijo de Carlos V fue (Fernando VII, Felipe II, Alfonso XIII).

____

EXERCISE C. **Indique si cada frase es cierta o falsa. Si es falsa, cámbiela para hacerla cierta.**

1. Los romanos reinaron hasta la invasión de los moros.

____

2. Se debe la introducción de la noria a los cartagineses.

____

3. Pelayo venció a los moros en la batalla de Sagunto.

____

4. Los españoles vencieron a los turcos en la batalla de Lepanto.

____

5. España llegó a tener muchas posesiones ultramarinas durante el reinado de Fernando e Isabel.

____

6. La «Armada Invencible» fue vencida por los ingleses en 1588.

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7. La primera república española duró cinco años.

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8. La decadencia de España se debió a muchas guerras, reyes débiles, y la expulsión de los judíos y los moros.

____

9. La Guerra Civil comenzó en 1936 y no terminó hasta 1941.

____

10. Hoy día el gobierno de España es una monarquía constitucional.

____

EXERCISE D. Complete las frases siguientes.

1. El español se deriva del ________________.
2. El día de la independencia española se celebra ________________.
3. Los cartagineses vencieron a los celtíberos en la batalla de ________________.
4. La Reconquista fue terminada por ________________.
5. Los moros introdujeron ________________ en España.
6. España sufrió una decadencia completa en el siglo ________________.
7. Los fenicios fundaron la ciudad de ________________.
8. Los españoles vencieron a los turcos en la batalla de ________________.
9. El héroe nacional de España es ________________.
10. Las guerras entre la reina Isabel y Carlos se llaman ________________.

Prueba 5-1

Comprensión del vocabulario 1

A. Esteban y unos amigos hablan sobre lo que quieren hacer este verano. Di qué trabajo busca cada uno, encerrando en un círculo la palabra que lo describe.

1. El periódico local de nuestra ciudad busca a alguien para repartir periódicos. El único requisito es tener una bicicleta. Voy a solicitar el puesto.
   a. gerente **b.** mensajero **c.** repartidor
2. Me gusta cuidar chicos y me llevo bien con ellos. Las horas de trabajo son flexibles. Voy a llamar para pedir una entrevista.
   a. entrenadora **b.** niñera **c.** clienta
3. Me gusta estar al sol y nado muy bien. Éste es el trabajo perfecto para mí.
   a. salvavida **b.** consejero **c.** recepcionista
4. Este verano voy a trabajar en un campamento con chicos de 8 a 12 años. Tengo que trabajar con ellos y ayudarlos a hacer ejercicio y divertirse sanamente.
   a. niñero **b.** consejero **c.** gerente
5. El año pasado trabajé en una tienda donde se reparan bicicletas. Como ahora tengo mucha experiencia, este año voy a estar encargado de la tienda.
   a. gerente **b.** mensajero **c.** dueño

B. Ricardo escribió una carta para solicitar un trabajo. Completa su carta subrayando la palabra entre paréntesis que mejor complete la frase.

Estimados Sres.:

Me llamo Ricardo Gamboa. Estoy en mi último año de la escuela secundaria y estoy muy interesado en trabajar en su *(computación / compañía)*. Soy un estudiante *(dedicado / entrometido)* en la escuela. Soy *(egoísta / responsable)* con mi trabajo y siempre *(cumplo / sueldo)* con mis tareas diarias. Tengo buenos *(conocimientos / requisitos)* de matemáticas y ciencias. Mi *(beneficio / habilidad)* con las computadoras es muy buena. Busco trabajo *(a tiempo completo / a tiempo parcial)*, entre 15 y 20 horas a la semana. Puedo trabajar horas *(puntuales / flexibles)*.

Sinceramente,

Ricardo Gamboa

C. Elena le habla a su amiga sobre un trabajo que obtuvo para el verano. Completa las frases con las palabras apropiadas de cada recuadro. No todas las palabras son necesarias.

solicitud de empleo	recepcionista	dueño	presentarme
a tiempo completo	anuncios clasificados	entrevista	cumplir

Como este verano quiero trabajar ____________________, 40 horas a la semana, compré el periódico para buscar un trabajo en los ____________________. Encontré varios, pero uno me llamó la atención. Es de ____________________ en el consultorio de un veterinario. Como tú sabes, a mí me encantan los animales. Llamé por teléfono y me dijeron que debía ____________________ a las 3:30 P.M. Allí tuve que llenar una ____________________ y después tuve la entrevista con el ____________________, el Dr. Ruiz.

puesto	requisito	computación	salario
fecha de nacimiento		experiencia	amable

Cuando fui a la entrevista, el Dr. Ruiz era un señor muy ____________________, pero no me creía mi edad, y tuve que darle una prueba de mi ____________________. Me preguntó si tenía experiencia en ____________________ y le dije que sí. Para el Dr. Ruiz, saber usar la computadora era lo más importante. Era el único ____________________. Me dio el ____________________ y me va a pagar un ____________________ de $400.00 por semana. Empiezo a trabajar a fines de junio. ¿Lo puedes creer?

Auténtico 3
Tema 5: Trabajo y comunidad

Lectura 1

1. C
2. C
3. C
4. A
5. D

Lectura 2

1. A
2. C
3. *Parte A:* D
   Parte B: C
4. A
5. D
6. A
7. B & D

Lectura 3

1. A
2. D
3. B
4. C
5. A
6. C
7. C

Integración de ideas
Escribir: *Answers will vary.*
Hablar y escuchar: *Answers will vary.*

Key

nas de Cuba, Guam, Puerto Rico, y las Islas Filipinas.

4. La Primera República fue establecida en 1873, y duró solamente un año. Alfonso XII estableció otra vez la monarquía. Su hijo, Alfonso XIII, fue rey de España hasta 1931. Durante su reinado, estableció una dictadura bajo Primo de Rivera. La segunda república fue establecida el 14 de abril de 1931.

5. En 1936 estalló una guerra civil que duró hasta 1939. El general Francisco Franco venció a las fuerzas de la república y estableció una dictadura. Franco rigió (*ruled*) el país hasta su muerte en 1975. El gobierno actual es una monarquía constitucional. En 1975, Juan Carlos I fue proclamado rey por las Cortes. En 1982, Felipe González Márquez llegó a ser presidente del Gobierno. A partir de 1985, España es miembro de la Comunidad Económica Europea (C.E.E.).

EXERCISE A. **A la izquierda de cada expresión de la lista A, escriba la letra de la expresión correspondiente de la lista B.**

	A	B
j	*1.* Pelayo	*a.* el Cid
e	*2.* los Reyes Católicos	*b.* Napoleón
a	*3.* Rodrigo Díaz de Vivar	*c.* rey de España hasta 1931
f	*4.* Felipe II	*d.* los Estados Unidos
b	*5.* Dos de Mayo	*e.* Fernando e Isabel
c	*6.* Alfonso XIII	*f.* Armada Invencible
d	*7.* guerra que ocurrió al fin del siglo XIX	*g.* monumento romano
g	*8.* puente de Alcántara	*h.* 14 de abril de 1931
h	*9.* la segunda república	*i.* dictador español
i	*10.* Francisco Franco	*j.* principió la Reconquista

EXERCISE B. **Escoja y subraye la palabra o expresión que complete correctamente cada frase.**

1. Los primeros habitantes de España fueron los (iberos, moros, romanos).

iberos

2. La segunda república española se estableció en el año (718, 1931, 1808).

1931

3. El acueducto de Segovia fue construido por los (fenicios, romanos, moros).

romanos

4. Muchas palabras españolas que principian con "al-" son de origen (romano, árabe, ibero).

árabe

5. Las guerras entre los moros y los españoles se llamaron (las Guerras Civiles, las Guerras Carlistas, la Reconquista).

La Reconquista

6. España perdió a Cuba, Puerto Rico, y las Filipinas en (la guerra civil, la guerra con los Estados Unidos, la guerra de independencia).

7. Pelayo venció a los moros en la batalla de (Covadonga, Lepanto, Sagunto).
Covadonga

8. Los visigodos vencieron a los (moros, romanos, fenicios). romanos

9. El dictador de España bajo Alfonso XIII fue (Francisco Franco, Primo de Rivera, Pelayo).
Franco

10. El hijo de Carlos V fue (Fernando VII, Felipe II, Alfonso XIII). Felipe II

EXERCISE C. Indique si cada frase es cierta o falsa. Si es falsa, cámbiela para hacerla cierta.

1. Los romanos reinaron hasta la invasión de los moros. F
Los visigodos

2. Se debe la introducción de la noria a los cartagineses. F
los moros

3. Pelayo venció a los moros en la batalla de Sagunto. F
Covadonga 718

4. Los españoles vencieron a los turcos en la batalla de Lepanto. C

5. España llegó a tener muchas posesiones ultramarinas durante el reinado de Fernando e Isabel. C

6. La «Armada Invencible» fue vencida por los ingleses en 1588. C

7. La primera república española duró cinco años. F
un año

8. La decadencia de España se debió a muchas guerras, reyes débiles, y ~~la expulsión de los judíos y los moros~~. F
Los altos impuestos

9. La Guerra Civil comenzó en 1936 y no terminó hasta 1941. F
hasta 1939

10. Hoy día el gobierno de España es una monarquía constitucional. C

EXERCISE D. **Complete las frases siguientes.**

1. El español se deriva del árabe y latín
2. El día de la independencia española se celebra 2 de mayo.
3. Los cartagineses vencieron a los celtíberos en la batalla de Sagunto.
4. La Reconquista fue terminada por Los Reyes Catolicos
5. Los moros introdujeron nuevas palabras en España.
6. España sufrió una decadencia completa en el siglo XVIII.
7. Los fenicios fundaron la ciudad de Cádiz.
8. Los españoles vencieron a los turcos en la batalla de Lepanto.
9. El héroe nacional de España es El Cid.
10. Las guerras entre la reina Isabel y Carlos se llaman Las Guerras Carlistas

GreatHearts Irving

Physics Remote Learning Packet

May 4-8, 2020

Course: 11 Physics
Teacher: Miss Weisse natalie.weisse@greatheartsirving.org
Resource: *Miss Weisse's Own Physics Textbook* — new pages found at the end of this packet

Weekly Plan:
Monday, May 4
- ☐ Read & Understand Notes on *Unit 8 Part 3 – Hooke's Law Background* (pages 51-54)
- ☐ Complete Pre-Lab Observations
- ☐ Prepare Your "Lab Book" Entry

Tuesday, May 5
- ☐ Take Data
- ☐ Create 4 Graphs
- ☐ Complete the Calculations and Analysis of All Four Graphs
- ☐ Email Miss Weisse with Questions

Wednesday, May 6
- ☐ Area Under the Curve Calculations for All Four Graphs
- ☐ Explanation of the Physical Significance of the Area Under the Curve
- ☐ Write Lab Conclusion
- ☐ Email Miss Weisse with Questions and to Ask for *Unit 8 Part 4* of *Miss Weisse's Own Physics Textbook*

Thursday, May 7
- ☐ Read & Understand Notes on *Unit 8 Part 4 - Hooke's Law & Elastic Energy* Notes
- ☐ Complete Unit 8 Worksheet 2 #1-4
- ☐ Email Miss Weisse with Questions and to Ask for Solutions

Friday, May 8
- ☐ Attend Office Hours at 9:30am!
- ☐ Review *Unit 8 Part 4 - Hooke's Law & Elastic Energy* Notes (pages 55-57)
- ☐ Complete Unit 8 Worksheet 2 #5-7
- ☐ Email Miss Weisse with Questions and to Ask for Solutions

Statement of Academic Honesty

I affirm that the work completed from the packet is mine and that I completed it independently.

I affirm that, to the best of my knowledge, my child completed this work independently

____
Student Signature

____
Parent Signature

Monday, May 4

- ➔ Read & Understand Notes on *Unit 8 Part 3 – Hooke's Law Background* (pages 51-54)
- ➔ Complete Pre-Lab Observations
    - ◆ Watch the short, two-minute video (https://safeYouTube.net/w/wMh8), observe the image of four spring scales below, and image the behavior of springs or rubber bands to answer the following three familiar questions on a sheet of paper, or on a google doc, to be turned in.
        - *What do you see?*
        - *What can you measure?*
        - *What can you change?*

    **As you observe the four spring scales, describe both the motion and behavior of the springs as they move *and* the difference between the four springs.

- ➔ Prepare Your "Lab Book" Entry **on a piece of paper**. *This assignment cannot be completed online.*
  - ◆ Write the Purpose of the lab (can be found in Unit 8 – Part 3... if you don't already know)
  - ◆ List your Independent Variable (length of spring), Dependent Variable (force of spring), and Constants (determine these yourself).

    *** Are our variable in your list of things we can measure and change?!
  - ◆ Create four data charts like the example chart below.

    NOTE! Extension is the length the spring stretched when a force is applied, not the total length of the spring. To calculate this, subtract the length at 0 Newtons force from each measure length.

    Extension = Current Length – Length @ 0 Newton Force

Green Scale			
Force (N)	**Length (cm)**	**Extension (cm)**	**Extension (m)**

Tuesday, May 5

- ➔ Take data using the images found at the end of this packet, after the textbook.
  - ◆ Don't forget our precision measurement rules! You should write a measurement that has one more decimal place than the ruler marks. So each of your measurements should have *2 decimal places*. Also, to take good data, zoom in on your screen to ~200%, then use a piece of paper to create a line between the bottom of the spring and the ruler markings.
- ➔ Create four graphs from your four sets of data. Each graph should take up *at least* ⅓ - ½ of the page.
- ➔ Complete the calculations and analysis for each of the four graphs

  As a reminder –
  - ◆ The calculations include the slope, the y-intercept, and the 5% rule for the y-intercept if your value is not zero.
  - ◆ The analysis includes what the slope tells us (focus on units!) and what the y-intercept tells us.

- Some helpful reminders are found on page 54 of *Miss Weisse's Own Physics Textbook*

Wednesday, May 6

- Now that you have an equation that describes the relationship between force exerted by the spring and the extension of the spring, it is time to relate this information to *ENERGY.* If you recall from our study of motion 1st semester, we found new, important physical properties by calculating the Area Under the Curve (AUC) of the velocity versus time graph and acceleration vs. time graph. We can do the same thing here with our spring force versus length graph. ***Calculate the AUC for all four graphs.***

- Explain the physics significance of the Area Under the Curve. ***What physical quantity does this measurement give us? Use the units to help puzzle it out.***

- Write a Lab Conclusion.

  - Restate the problem and summarize how the data was taken.
  - State the relationship between the two variables, explain what the slope means, and state what the y-intercept tells us.
  - Compare the four different springs, giving numerical data.
  - Explain what the area under the curve tells us.
  - If your data has problems, provide possible explanations.
  - Provide an answer to the original problem.

  ***Today's work can be done on the same piece of paper as Tuesday's lab work.

- Email Miss Weisse with Questions and to Ask for *Unit 8 Part 4* of *Miss Weisse's Own Physics Textbook*

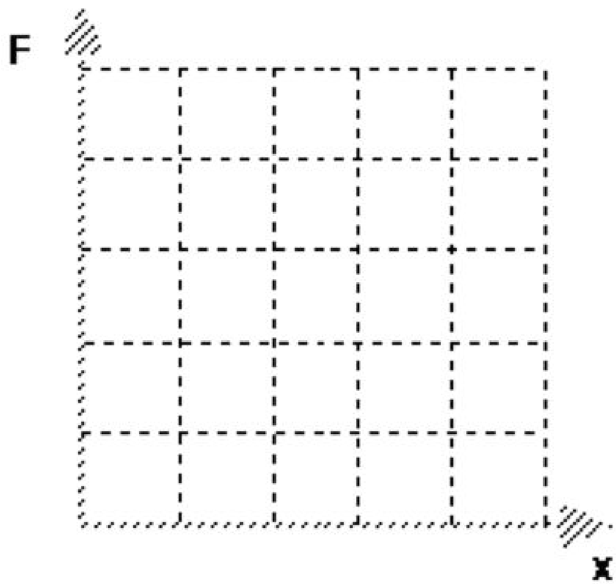
Thursday, May 7

- ➔ If you have not done so yet, turn in your lab and request *Unit 8 Part 4* from Miss Weisse via Email!
- ➔ Read & Understand Notes on *Unit 8 Part 4 - Hooke's Law & Elastic Energy*
- ➔ On a sheet of paper with a full heading, complete Unit 8 Worksheet 2 #1-4

Energy Storage and Transfer Model Worksheet 2 (part #1-4):

Hooke's Law and Elastic Energy

Directions: Suppose one lab group found that F = 1000 N/m (Δx). Construct a graphical representation of force vs. displacement. (Hint: make the maximum displacement 0.25 m.)

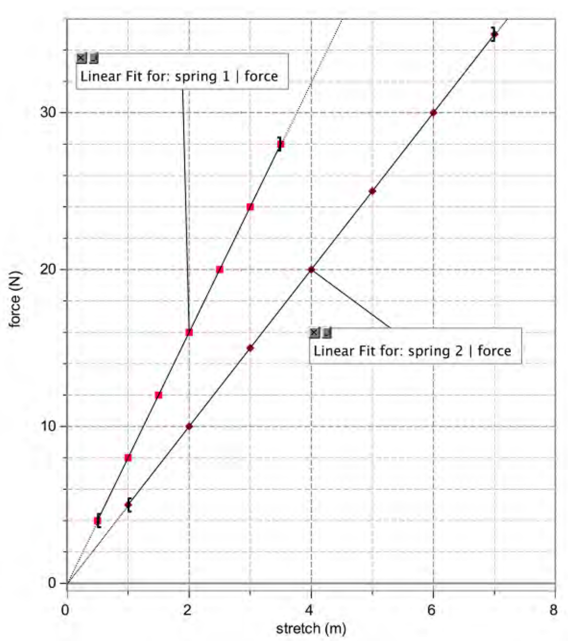


1. Graphically determine the amount of energy stored while stretching the spring described above from x = 0 to x = 10. cm.

2. Graphically determine the amount of energy stored while stretching the spring described above from x = 15 to x = 25 cm.

Directions: The graph below was made from data collected during an investigation of the relationship between the amounts two different springs stretched when different forces were applied.

3. Determine the spring constant for each spring.



4. For each spring, compare:

a. the amount of force required to stretch the spring 3.0 m.

b. the E_{el} stored in each spring when stretched 3.0m.

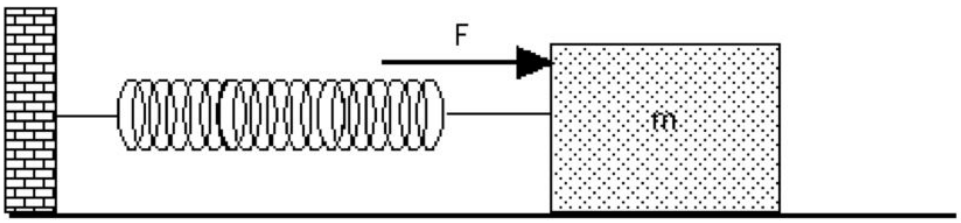
Friday, May 8

- ➔ Attend Office Hours at 9:30am! I want to talk to you and help you!
- ➔ Review *Unit 8 Part 4 - Hooke's Law & Elastic Energy* Notes
- ➔ On a sheet of paper with a full heading, complete Unit 8 Worksheet 2 #5-7

Energy Storage and Transfer Model Worksheet 2 (#5-7):
Hooke's Law and Elastic Energy

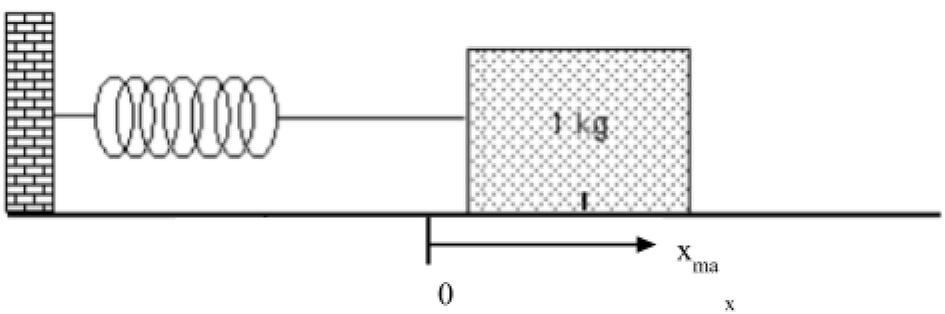
5. Determine the amount that spring 2 needs to be stretched in order to store 24 joules of energy.

6. The spring below has a spring constant of 10. N/m. If the block is pulled 0.30 m horizontally to the



right, and held motionless, what force does the spring exert on the block? Sketch a force diagram for the mass as you hold it still. (Assume a frictionless surface.)

7. The spring below has a spring constant of 20. N/m. The μ_s between the box and the surface is 0.40.



a. The box is pushed to the right, then released. Draw a force diagram for the box above when the spring is stretched, yet the box is stationary.

b. What is the maximum distance that the spring can be stretched from equilibrium before the box begins to slide back?

c. Do pie chart analysis for this situation, when the spring is stretched beyond its maximum (from part b above) so it slides back, and then the box oscillates back and forth until it comes to a stop. Assume your system includes the spring, box, and table top.

Unit 8

Part 3 - Hooke's Law Background

Many of the problems we did last week had to do with elastic energy. Elastic energy is found in things that can regain their shape after being deformed by some force. Obvious examples of this are the a stretched rubberband, a trampoline when someone jumps on it, or a wind-up toy's mechanism when it gets wound up.

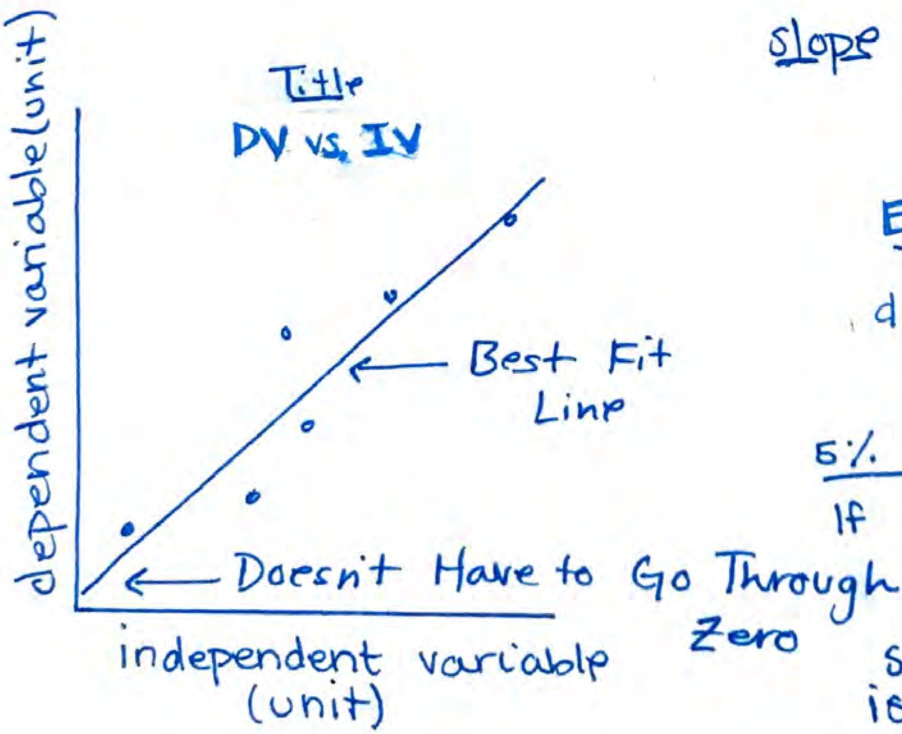
But, in fact, the elastic properties of matter are involved in many physical phenomena. When matter is deformed (compressed, twisted, stretched, etc) and the deforming forces are sufficiently small, the material will return to its original shape when the deforming forces are removed. Steel wires, concrete columns, metal beams and rods, and other material objects can also undergo elastic deformations, but they are often too small for us to see. For many materials, it is approximately true that when the material is stretched or compressed, the resisting or restoring force that tends to return the material to its original shape is related to the amount of the deformation but points in the opposite direction opposite to the stretch or compression — Newton's Third Law! So the more force that is applied to a spring, the more the spring stretches, causing the spring to apply a larger force.

The English physicist, Robert Hooke, recorded the relationship between a spring and a mass attached vertically to the spring (so the gravitational force on the mass is pulling down on the spring). This idealized behavior of matter is called HOOKE'S LAW. In the following lab, you will determine the relationship between the force used to deform (stretch) a spring and the amount that the spring has been attached. Unlike Hooke's original labs, we are using spring scales laid horizontally on a surface. The spring scale shows a gradient of forces along the column that houses the scale. As one end of the scale is pulled (with the other end in place), a force is exerted on the spring so that it extends in length and pulls back with equal force. As you pull you feel the spring pulling back. Think about what happens if you just release the spring after pulling it – what happens and what causes what happens?

THE POST-LAB SECTION OF
Miss Weisse's Own Physics Textbook
WILL BE MADE AVAILABLE TO YOU AFTER
YOU TURN IN YOUR LAB.

Helpful Reminders

MEASUREMENT	EQUATION(S)	UNIT(S)	VARIABLE	UNIT BREAKDOWN
speed/ velocity	$=\frac{\Delta x}{\Delta t}$	$\frac{\text{meter}}{\text{second}}$	m/s	na
acceleration	$=\frac{\Delta v}{t}$	meter/second^2	$\frac{m}{s^2}$	$\frac{\Delta v}{t} \Rightarrow \frac{(\frac{m}{s})}{s} = \frac{m}{s \cdot s} = \frac{m}{s^2}$
force	$= m \cdot \vec{a}$	Newton	N	$m \cdot a = kg \cdot (\frac{m}{s^2}) = \frac{kg \cdot m}{s^2}$
momentum	$= m \cdot \vec{v}$	kilogram· meterper second	$\frac{kg \cdot m}{s}$	$m \cdot v = kg(\frac{m}{s}) = \frac{kg \cdot m}{s}$
energy	$= F \cdot d$ (one of many equations)	Joule	J	$= F \cdot d = N \cdot m = \frac{kg \cdot m}{s^2} \cdot m = \frac{kg \cdot m^2}{s^2}$



slope $= m = \frac{\Delta(DV)}{\Delta(IV)}$

Equation of a Line

$dv = m(iv) + b$

5% Rule

If $b < (.05)(\text{max-dv value})$ then we can say the y-intercept is negligable.

SET UP

***********************to make your measurements you will want to zoom in on your computer screen (~200%)*********************

FROM THE FRONT: The spring scale is being anchored in place with a pen perpendicular to the board, so the whole scale cannot be pulled forward when the hook is pulled forward to extend the spring.	**FROM THE TOP:** Again, the spring scale is anchored by the pen and the top of the spring is aligned with the zero mark of the ruler. The ruler remains stationary throughout the measurements.

GREEN SPRING SCALE (VIDEO)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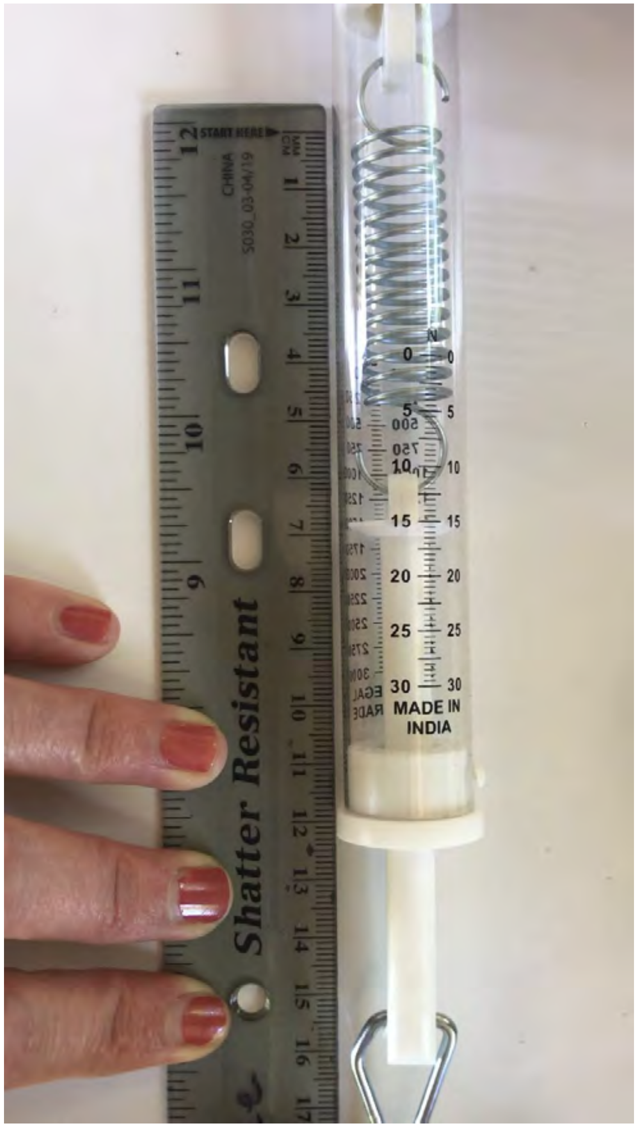
WHITE SPRING SCALE (VIDEO)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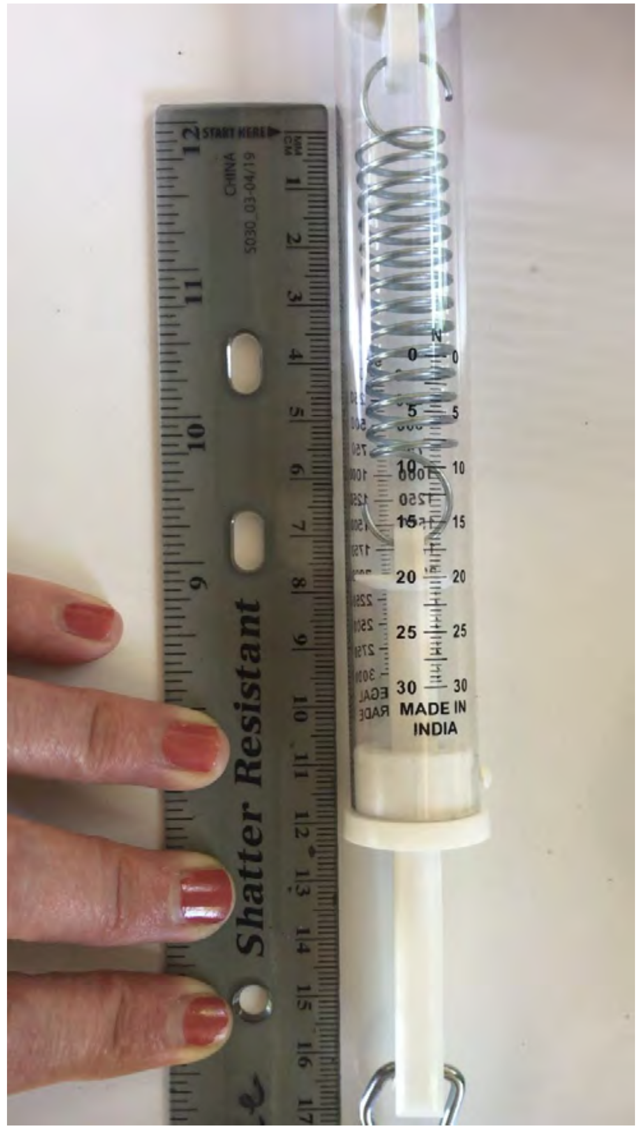
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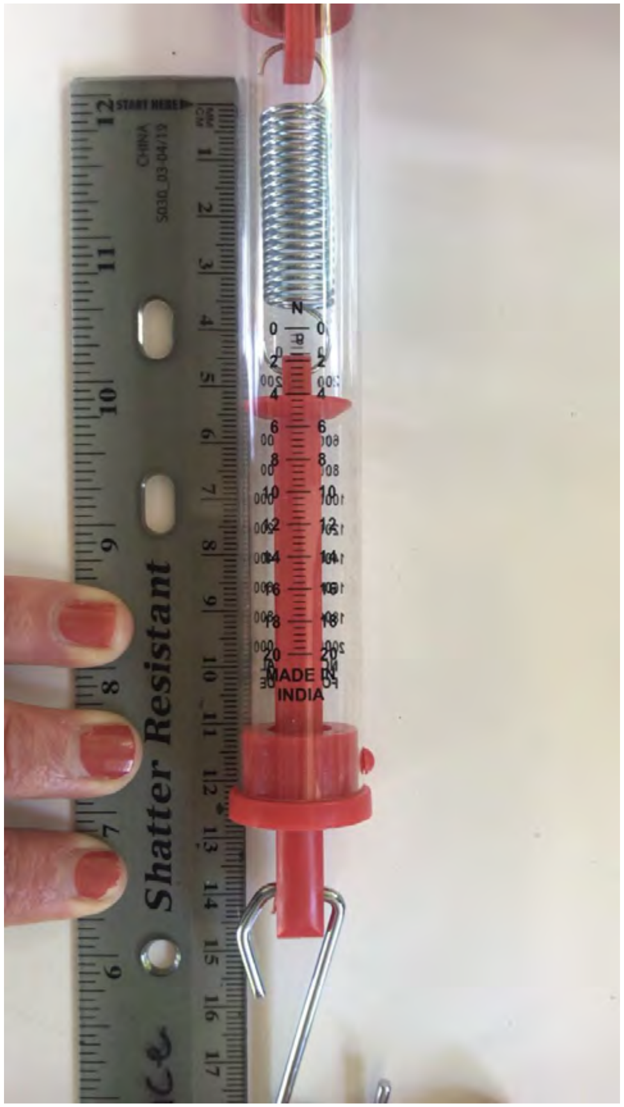
RED SPRING SCALE (VIDEO)

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BROWN SPRING SCALE (VIDEO)


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