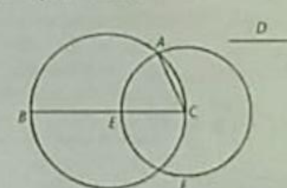
Book IV

Definitions

1. A rectilineal figure is said to be *inscribed in a rectilineal figure* when the respective angles of the inscribed figure lie on the respective sides of that in which it is inscribed.
2. Similarly a figure is said to be *circumscribed about a figure* when the respective sides of the circumscribed figure pass through the respective angles of that about which it is circumscribed.
3. A rectilineal figure is said to be *inscribed in a circle* when each angle of the inscribed figure lies on the circumference of the circle.
4. A rectilineal figure is said to be *circumscribed about a circle*, when each side of the circumscribed figure touches the circumference of the circle.
5. Similarly a circle is said to be *inscribed in a figure* when the circumference of the circle touches each side of the figure in which it is inscribed.
6. A circle is said to be *circumscribed about a figure* when the circumference of the circle passes through each angle of the figure about which it is circumscribed.
7. A straight line is said to be *fitted into a circle* when its extremities are on the circumference of the circle.

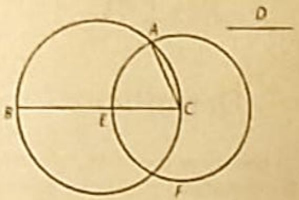
Proposition 1

Into a given circle to fit a straight line equal to a given straight line which is not greater than the diameter of the circle.



Let *ABC* be the given circle, and *D* the given straight line not greater than the diameter of the circle;
thus it is required to fit into the circle *ABC* a straight line equal to the straight line *D*.

Let a diameter *BC* of the circle *ABC* be drawn.



Then, if *BC* is equal to *D*, that which was enjoined will have been done; for *BC* has been fitted into the circle *ABC* equal to the straight line *D*.

But, if *BC* is greater than *D*, let *CE* be made equal to *D*,
and with centre *C* and distance *CE* let the circle *EAF* be described;
let *CA* be joined.

Then, since the point *C* is the centre of the circle *EAF*,

CA is equal to *CE*.

But *CE* is equal to *D*;

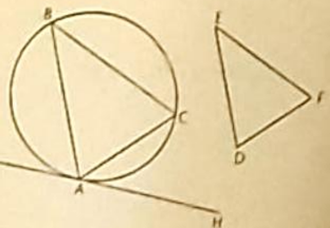
therefore *D* is also equal to *CA*.

Therefore into the given circle *ABC* there has been fitted *CA* equal to the given straight line *D*.

Q.E.F.

Proposition 2

In a given circle to inscribe a triangle equiangular with a given triangle.



Let *ABC* be the given circle, and *DEF* the given triangle;
thus it is required to inscribe in the circle *ABC* a triangle equiangular with the triangle *DEF*.

Let *GH* be drawn touching the circle *ABC* at *A*; [III. 16, Por.]
on the straight line *AH*, and at the point *A* on it, let the angle *HAC* be constructed equal to the angle *DEF*,
and on the straight line *AG*, and at the point *A* on it, let the angle *GAB* be constructed equal to the angle *DFE*; [I. 23]
let *BC* be joined.

Then, since a straight line *AH* touches the circle *ABC*,
and from the point of contact at *A* the straight line *AC* is drawn across in the circle,
therefore the angle *HAC* is equal to the angle *ABC* in the alternate segment of the circle. [III. 32]

But the angle *HAC* is equal to the angle *DEF*,

therefore the angle *ABC* is also equal to the angle *DEF*.

For the same reason

the angle *ACB* is also equal to the angle *DFE*;

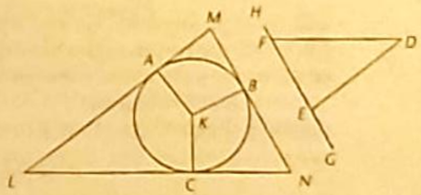
therefore the remaining angle *BAC* is also equal to the remaining angle *EDF*. [I. 32]

Therefore in the given circle there has been inscribed a triangle equiangular with the given triangle.

Q.E.F.

Proposition 3

About a given circle to circumscribe a triangle equiangular with a given triangle.



Let *ABC* be the given circle, and *DEF* the given triangle;
thus it is required to circumscribe about the circle *ABC* a triangle equiangular with the triangle *DEF*.

Let *EF* be produced in both directions to the points *G*, *H*,
let the centre *K* of the circle *ABC* be taken, [III. 1]
and let the straight line *KB* be drawn across at random;
on the straight line *KB*, and at the point *K* on it, let the angle *BKA* be constructed equal to the angle *DEG*,
and the angle *BKC* equal to the angle *DFH*; [I. 23]
and through the points *A*, *B*, *C* let *LAM*, *MBN*, *NCL* be drawn touching the circle *ABC*. [III. 16, Por.]

Now, since *LM*, *MN*, *NL* touch the circle *ABC* at the points *A*, *B*, *C*,
and *KA*, *KB*, *KC* have been joined from the centre *K* to the points *A*, *B*, *C*,

therefore the angles at the points *A*, *B*, *C* are right. [III. 18]

And, since the four angles of the quadrilateral *AMBK* are equal to four right angles, inasmuch as *AMBK* is in fact divisible into two triangles,
and the angles *KAM*, *KBM* are right,

therefore the remaining angles *AKB*, *AMB* are equal to two right angles.

But the angles *DEG*, *DEF* are also equal to two right angles; [I. 13]

therefore the angles *AKB*, *AMB* are equal to the angles *DEG*, *DEF*,

of which the angle *AKB* is equal to the angle *DEG*;
therefore the angle *AMB* which remains is equal to the angle *DEF* which remains.

Similarly it can be proved that the angle *LNB* is also equal to the angle *DFE*;

therefore the remaining angle *MLN* is equal to the angle *EDF*. [I. 32]

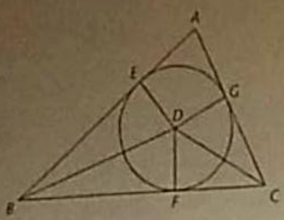
Therefore the triangle *LMN* is equiangular with the triangle *DEF*;
and it has been circumscribed about the circle *ABC*.

Therefore about a given circle there has been circumscribed a triangle equiangular with the given triangle.

Q.E.F.

Proposition 4

In a given triangle to inscribe a circle.



Let ABC be the given triangle;
thus it is required to inscribe a circle in the triangle ABC.

Let the angles ABC, ACB be bisected by the straight lines BD, CD, [I. 9]
and let these meet one another at the point D;
from D let DE, DF, DG be drawn perpendicular to the straight lines AB, BC, CA.

Now, since the angle ABD is equal to the angle CBD,
and the right angle BED is also equal to the right angle BFD,
EBD, FBD are two triangles having two angles equal to two angles and one side equal to one side, namely that subtending one of the equal angles, which is BD common to the triangles;
therefore they will also have the remaining sides equal to the remaining sides; [I. 26]

therefore DE is equal to DF.

For the same reason
DG is also equal to DF.

Therefore the three straight lines DE, DF, DG are equal to one another;
therefore the circle described with centre D and distance one of the straight lines DE, DF, DG will pass also through the remaining points, and will touch the straight lines AB, BC, CA, because the angles at the points E, F, G are right.

For, if it cuts them, the straight line drawn at right angles to the diameter of the circle from its extremity will be found to fall within the circle:
which was proved absurd; [III. 16]
therefore the circle described with centre D and distance one of the straight lines DE, DF, DG will not cut the straight lines AB, BC, CA;
therefore it will touch them, and will be the circle inscribed in the triangle ABC. [IV. Def. 5]

Let it be inscribed, as FGE.

Therefore in the given triangle ABC the circle EFG has been inscribed.

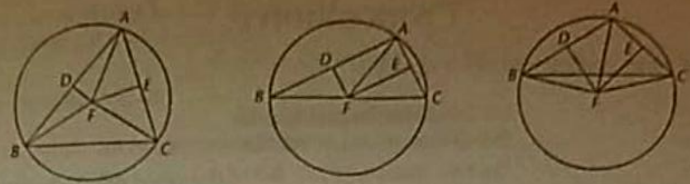
Q.E.F.

Proposition 5

About a given triangle to circumscribe a circle.

Let ABC be the given triangle;
thus it is required to circumscribe a circle about the given triangle ABC.

Let the straight lines AB, AC be bisected at the points D, E, [I. 10]
and from the points D, E let DF, EF be drawn at right angles to AB, AC;



they will then meet within the triangle ABC, or on the straight line BC, or outside BC.

First let them meet within at F, and let FB, FC, FA be joined.

Then, since AD is equal to DB,
and DF is common and at right angles,
therefore the base AF is equal to the base FB. [I. 4]

Similarly we can prove that CF is also equal to AF;
so that FB is also equal to FC;
therefore the three straight lines FA, FB, FC are equal to one another.

Therefore the circle described with centre F and distance one of the straight lines FA, FB, FC will pass also through the remaining points, and the circle will have been circumscribed about the triangle ABC.

Let it be circumscribed, as ABC.

Next, let DF, EF meet on the straight line BC at F, as is the case in the second figure; and let AF be joined.

Then, similarly, we shall prove that the point F is the centre of the circle circumscribed about the triangle ABC.

Again, let DF, EF meet outside the triangle ABC at F, as is the case in the third figure, and let AF, BF, CF be joined.

Then again, since AD is equal to DB,
and DF is common and at right angles,
therefore the base AF is equal to the base BF. [I. 4]

Similarly we can prove that CF is also equal to AF;
so that BF is also equal to FC;

therefore the circle described with centre F and distance one of the straight lines FA, FB, FC will pass also through the remaining points, and will have been circumscribed about the triangle ABC.

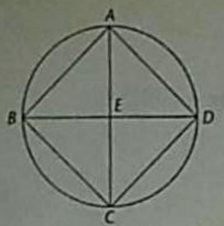
Therefore about the given triangle a circle has been circumscribed.

Q.E.F.

And it is manifest that, when the centre of the circle falls within the triangle, the angle BAC, being in a segment greater than the semicircle, is less than a right angle;
when the centre falls on the straight line BC, the angle BAC, being in a semicircle, is right;
and when the centre of the circle falls outside the triangle, the angle BAC, being in a segment less than the semicircle, is greater than a right angle. [III. 31]

Proposition 6

In a given circle to inscribe a square.



Let *ABCD* be the given circle;
thus it is required to inscribe a square in the circle *ABCD*.

Let two diameters *AC*, *BD* of the circle *ABCD* be drawn at right angles to one another, and let *AB*, *BC*, *CD*, *DA* be joined.

Then, since *BE* is equal to *ED*, for *E* is the centre, and *EA* is common and at right angles,
therefore the base *AB* is equal to the base *AD*. [I. 4]

For the same reason
each of the straight lines *BC*, *CD* is also equal to each of the straight lines *AB*, *AD*;
therefore the quadrilateral *ABCD* is equilateral.

I say next that it is also right-angled.

For, since the straight line *BD* is a diameter of the circle *ABCD*,
therefore *BAD* is a semicircle;
therefore the angle *BAD* is right. [III. 31]

For the same reason
each of the angles *ABC*, *BCD*, *CDA* is also right;
therefore the quadrilateral *ABCD* is right-angled.

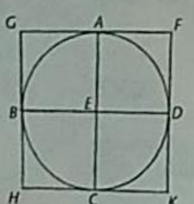
But it was also proved equilateral;
therefore it is a square; [I. Def. 22]
and it has been inscribed in the circle *ABCD*.

Therefore in the given circle the square *ABCD* has been inscribed.

Q.E.F.

Proposition 7

About a given circle to circumscribe a square.



Let *ABCD* be the given circle;
thus it is required to circumscribe a square about the circle *ABCD*.

Let two diameters *AC*, *BD* of the circle *ABCD* be drawn at right angles to one another, and through the points *A*, *B*, *C*, *D* let *FG*, *GH*, *HK*, *KF* be drawn touching the circle *ABCD*. [III. 16, Por.]

Then, since *FG* touches the circle *ABCD*,
and *EA* has been joined from the centre *E* to the point of contact at *A*,
therefore the angles at *A* are right. [III. 18]

For the same reason
the angles at the points *B*, *C*, *D* are also right.

Now, since the angle *AEB* is right,
and the angle *EBG* is also right,
therefore *GH* is parallel to *AC*. [I. 28]

For the same reason
AC is also parallel to *FK*,
so that *GH* is also parallel to *FK*. [I. 30]

Similarly we can prove that
each of the straight lines *GF*, *HK* is parallel to *BED*.

Therefore *GK*, *GC*, *AK*, *FB*, *BK* are parallelograms;
therefore *GF* is equal to *HK*, and *GH* to *FK*. [I. 34]

And, since *AC* is equal to *BD*,
and *AC* is also equal to each of the straight lines *GH*, *FK*,
while *BD* is equal to each of the straight lines *GF*, *HK*, [I. 34]
therefore the quadrilateral *FGHK* is equilateral.

I say next that it is also right-angled.

For, since *GBEA* is a parallelogram,
and the angle *AEB* is right,
therefore the angle *AGB* is also right. [I. 34]

Similarly we can prove that
the angles at *H*, *K*, *F* are also right.

Therefore *FGHK* is right-angled.

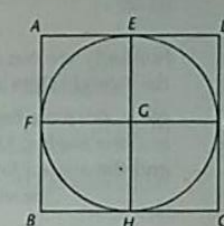
But it was also proved equilateral;
therefore it is a square;
and it has been circumscribed about the circle *ABCD*.

Therefore about the given circle a square has been circumscribed.

Q.E.F.

Proposition 8

In a given square to inscribe a circle.

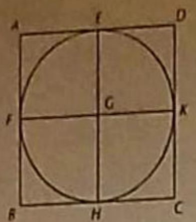


Let *ABCD* be the given square;
thus it is required to inscribe a circle in the given square *ABCD*.

Let the straight lines *AD*, *AB* be bisected at the points *E*, *F* respectively, [I. 10]
through *E* let *EH* be drawn parallel to either *AB* or *CD*, and through *F* let *FK* be drawn parallel to either *AD* or *BC*; [I. 31]
therefore each of the figures *AK*, *KB*, *AH*, *HD*, *AG*, *GC*, *BG*, *GD* is a parallelogram, and their opposite sides are evidently equal. [I. 34]

Now, since *AD* is equal to *AB*,
and *AE* is half of *AD*, and *AF* half of *AB*,
therefore *AE* is equal to *AF*,
so that the opposite sides are also equal;

therefore *FG* is equal to *GE*.



Similarly we can prove that each of the straight lines *GH, GK* is equal to each of the straight lines *FG, GE*; therefore the four straight lines *GE, GF, GH, GK* are equal to one another.

Therefore the circle described with centre G and distance one of the straight lines GE, GF, GH, GK will pass also through the remaining points.

And it will touch the straight lines *AB, BC, CD, DA*, because the angles at *E, F, H, K* are right.

For, if the circle cuts *AB, BC, CD, DA*, the straight line drawn at right angles to the diameter of the circle from its extremity will fall within the circle: which was proved absurd; [III. 16]
therefore the circle described with centre G and distance one of the straight lines *GE, GF, GH, GK* will not cut the straight lines *AB, BC, CD, DA*.

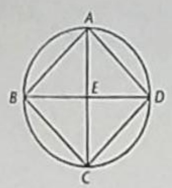
Therefore it will touch them, and will have been inscribed in the square *ABCD*.

Therefore in the given square a circle has been inscribed.

Q.E.F.

Proposition 9

About a given square to circumscribe a circle.



Let *ABCD* be the given square;
thus it is required to circumscribe a circle about the square *ABCD*.

For let *AC, BD* be joined, and let them cut one another at *E*.

Then, since *DA* is equal to *AB*, and *AC* is common,
therefore the two sides *DA, AC* are equal to the two sides *BA, AC*;
and the base *DC* is equal to the base *BC*;
therefore the angle *DAC* is equal to the angle *BAC*. [I. 8]

Therefore the angle *DAB* is bisected by *AC*.

Similarly we can prove that each of the angles *ABC, BCD, CDA* is bisected by the straight lines *AC, DB*.

Now, since the angle *DAB* is equal to the angle *ABC*,
and the angle *EAB* is half the angle *DAB*,
and the angle *EBA* half the angle *ABC*,
therefore the angle *EAB* is also equal to the angle *EBA*;
so that the side *EA* is also equal to *EB*. [I. 6]

Similarly we can prove that each of the straight lines *EA, EB* is equal to each of the straight lines *EC, ED*.

Therefore the four straight lines *EA, EB, EC, ED* are equal to one another.

Therefore the circle described with centre *E* and distance one of the straight lines *EA, EB, EC, ED* will pass also through the remaining points;

and it will have been circumscribed about the square *ABCD*.

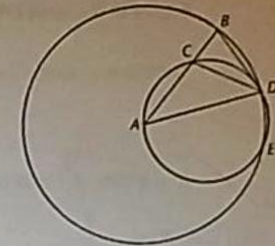
Let it be circumscribed, as *ABCD*.

Therefore about the given square a circle has been circumscribed.

Q.E.F.

Proposition 10

To construct an isosceles triangle having each of the angles at the base double of the remaining one.



Let any straight line *AB* be set out, and let it be cut at the point *C* so that the rectangle contained by *AB, BC* is equal to the square on *CA*; [II. 11]
with centre *A* and distance *AB* let the circle *BDE* be described,
and let there be fitted in the circle *BDE* the straight line *BD* equal to the straight line *AC* which is not greater than the diameter of the circle *BDE*. [IV. 1]

Let *AD, DC* be joined, and let the circle *ACD* be circumscribed about the triangle *ACD*. [IV. 5]

Then, since the rectangle *AB, BC* is equal to the square on *AC*,
and *AC* is equal to *BD*,
therefore the rectangle *AB, BC* is equal to the square on *BD*.

And, since a point *B* has been taken outside the circle *ACD*,
and from *B* the two straight lines *BA, BD* have fallen on the circle *ACD*, and one of them cuts it, while the other falls on it,
and the rectangle *AB, BC* is equal to the square on *BD*,
therefore *BD* touches the circle *ACD*. [III. 37]

Since, then, *BD* touches it, and *DC* is drawn across from the point of contact at *D*,
therefore the angle *BDC* is equal to the angle *DAC* in the alternate segment of the circle. [III. 32]

Since, then, the angle *BDC* is equal to the angle *DAC*,
let the angle *CDA* be added to each;
therefore the whole angle *BDA* is equal to the two angles *CDA, DAC*.

But the exterior angle *BCD* is equal to the angles *CDA, DAC*; [I. 32]
therefore the angle *BDA* is also equal to the angle *BCD*.

But the angle *BDA* is equal to the angle *CBD*, since the side *AD* is also equal to *AB*; [I. 5]
so that the angle *DBA* is also equal to the angle *BCD*.

Therefore the three angles *BDA, DBA, BCD* are equal to one another.

And, since the angle *DBC* is equal to the angle *BCD*,
the side *BD* is also equal to the side *DC*. [I. 6]
But *BD* is by hypothesis equal to *CA*;
therefore *CA* is also equal to *CD*,

Now, since the square on *EC* is equal to the square on *CA*,
the squares on *EC*, *CA* are double of the square on *CA*.

But the square on *EA* is equal to the squares on *EC*, *CA*; [I. 47]
therefore the square on *EA* is double of the square on *AC*. [C.N. 1]

Again, since *FG* is equal to *EF*,
the square on *FG* is also equal to the square on *FE*;
therefore the squares on *GF*, *FE* are double of the square on *EF*.

But the square on *EG* is equal to the squares on *GF*, *FE*; [I. 47]
therefore the square on *EG* is double of the square on *EF*.

And *EF* is equal to *CD*; [I. 34]
therefore the square on *EG* is double of the square on *CD*.

But the square on *EA* was also proved double of the square on *AC*;
therefore the squares on *AE*, *EG* are double of the squares on *AC*, *CD*.

And the square on *AG* is equal to the squares on *AE*, *EG*; [I. 47]
therefore the square on *AG* is double of the squares on *AC*, *CD*.

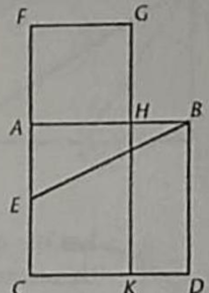
But the squares on *AD*, *DG* are equal to the square on *AG*; [I. 47]
therefore the squares on *AD*, *DG* are double of the squares on *AC*, *CD*.

And *DG* is equal to *DB*;
therefore the squares on *AD*, *DB* are double of the squares on *AC*, *CD*.

Therefore etc. Q.E.D.

Proposition 11

To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.



Let *AB* be the given straight line;
thus it is required to cut *AB* so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.

For let the square *ABDC* be described on *AB*; [I. 46]
let *AC* be bisected at the point *E*, and let *BE* be joined;
let *CA* be drawn through to *F*, and let *EF* be made equal to *BE*;
let the square *FH* be described on *AF*, and let *GH* be drawn through to *K*.

I say that *AB* has been cut at *H* so as to make the rectangle contained by *AB*, *BH* equal to the square on *AH*.

For, since the straight line *AC* has been bisected at *E*, and *FA* is added to it, the rectangle contained by *CF*, *FA* together with the square on *AE* is equal to the square on *EF*. [II. 6]

But *EF* is equal to *EB*;
therefore the rectangle *CF*, *FA* together with the square on *AE* is equal to the square on *EB*.

But the squares on *BA*, *AE* are equal to the square on *EB*, for the angle at *A* is right; [I. 47]