Remote Learning Packet

There is no need to submit this packet at the end of the week. Enjoy your summer break!

Week 9: May 25-29, 2020
Course: Precalculus
Teacher(s): Mr. Simmons

Monday, May 25

Happy Memorial Day! No School!

Tuesday, May 26 - Friday May 29

It's been wonderful teaching you all precalculus this year! For this week, we have a single challenging problem. There are three parts to the following document:

1. The problem.
2. A hint.
3. A correct solution.

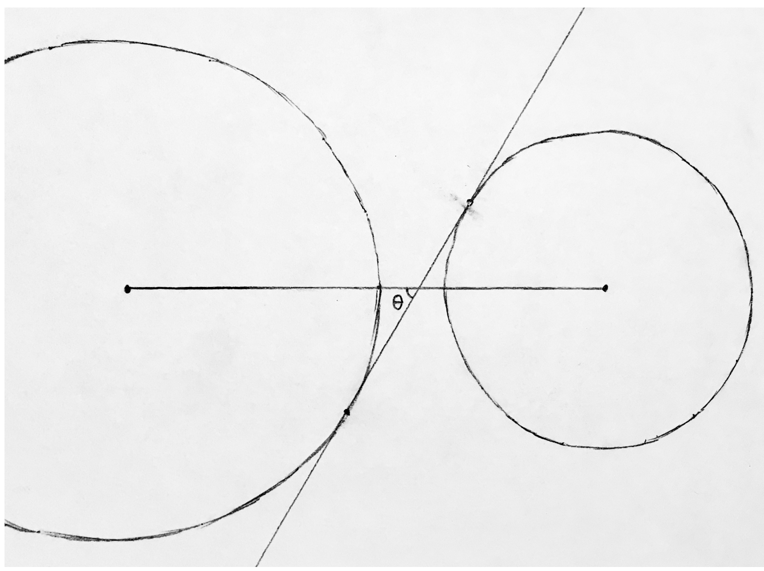
I encourage everyone first to read only the problem and attempt to solve it. Once you've given it your best effort, if you're stuck, take a look at the hint and continue attempting the problem. If you find that you are still unable to solve it, read through the correct solution, and make sure you understand why this solution is correct. I also encourage you to come up with other interesting problems and try to solve them yourselves! Notice how this problem doesn't take long to state, but it is quite complex to solve.

I hope that you have enjoyed this year. Enjoy your summer when it comes! If you have any fun summer plans or any parting words, feel free to email me!

A Challenging Problem

Precalculus
Mr. Simmons

Problem. Two circles are drawn with a distance of 2 between their centers. One of the cirlces has radius 1 and the other radius $\sqrt{3}-1$ (which is approximately 0.7). A line is drawn tangent to both, as in the diagram below. Angle θ is formed by the tangent line and the line between the centers, as shown in the diagram. Determine angle θ.

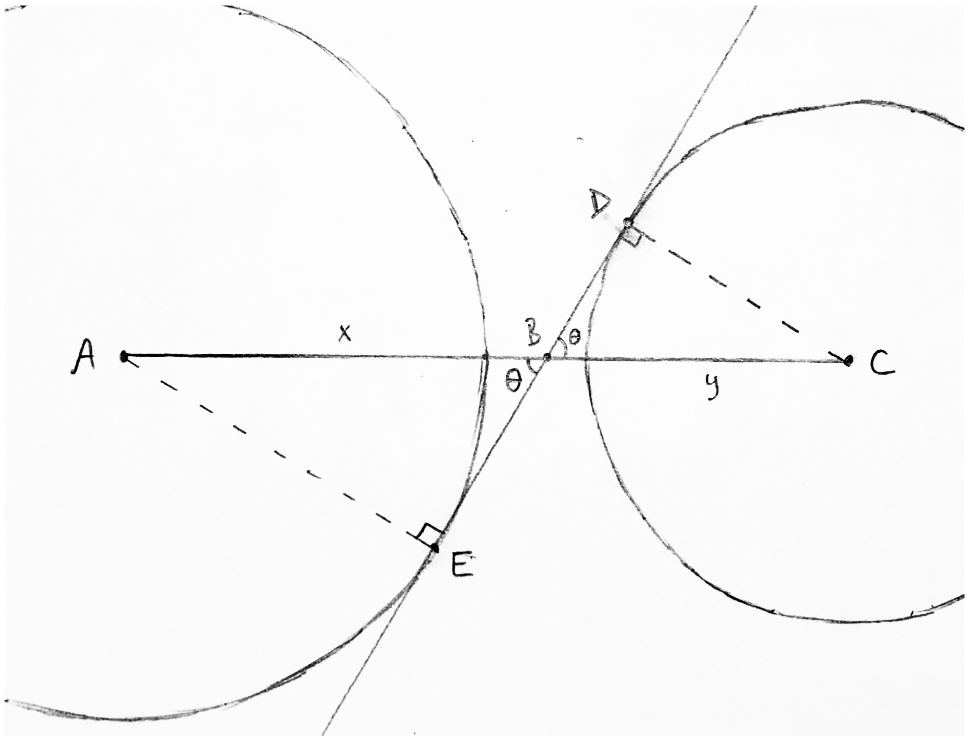


You have enough information to get started, but if you would like a hint, look at the next page.

Hint: A tangent line touches a circle at exactly one point. Call that point P, and call the center of the circle C. Then the tangent line is perpendicular to the line CP.

The complete solution starts on the next page.

Solution. At whatever point a tangent line touches a circle, it will be perpendicular to the radius line at that point. Labeling points in the diagram as below, this means that ED is perpendicular to both AE and CD.



Next, we observe that the vertical angle of θ is congruent to it, so we can label it θ, too. We have $AC = 2$, $AE = 1$, and $CD = \sqrt{3} - 1$, and for convenience we let $x = AB$ and $y = BC$. The diagram drawn thus, finding θ is a matter of relating it to the sides of these two triangles and solving for θ. The trigonometric function sine gives us

$$\sin\theta = \frac{AE}{AB} = \frac{1}{x} \quad \text{and} \quad \sin\theta = \frac{CD}{CB} = \frac{\sqrt{3}-1}{y}.$$

Finding x or y will then let us see whether $\sin\theta$ is a known sine value, which could tell us what θ is. So let's solve for one of those. From the above equations, it follows that

$$\frac{1}{x} = \frac{\sqrt{3}-1}{y}.$$

And clearly $AB + BC = AC$, so we also have

$$x + y = 2.$$

These two equations form a two-variable system of equations, which we now solve for x:

$$\begin{aligned}
\frac{1}{x} &= \frac{\sqrt{3}-1}{y} \\
\frac{1}{\cancel{x}}\cancel{x}y &= \frac{\sqrt{3}-1}{\cancel{y}}x\cancel{y} \\
y &= x\left(\sqrt{3}-1\right),
\end{aligned}$$

and so

$$\begin{aligned}
x + y &= 2 \\
x + x\left(\sqrt{3}-1\right) &= 2 \\
x\left(1+\sqrt{3}-1\right) &= 2 \\
x\sqrt{3} &= 2 \\
x &= \frac{2}{\sqrt{3}}.
\end{aligned}$$

How does this help us find θ? We go back to one of the expressions containing it:

$$\begin{aligned}
\sin\theta &= \frac{1}{x} \\
&= \frac{1}{\left(\frac{2}{\sqrt{3}}\right)} \\
&= \frac{\sqrt{3}}{2}.
\end{aligned}$$

We can see that θ is acute, and there is only one value for θ between 0 and $\pi/2$ that yields $\sin\theta = \sqrt{3}/2$, so we now know $\theta = \pi/3$.