GreatHearts
Irving

Remote Learning Packet

Please submit scans of written work in Google Classroom at the end of the week.

Week 7: May 11-15, 2020
Course: Calculus I
Teacher(s): Mr. Simmons

Weekly Plan:

Monday, May 11
- ☐ Story time!
- ☐ Check Practice Problems from Week 6
- ☐ Read Example Responses for Week 6
- ☐ Week 7: Practice Problems

Tuesday, May 12
- ☐ Week 7: Conceptual Questions 1-4

Wednesday, May 13
- ☐ Week 7: Conceptual Questions 5-6

Thursday, May 14
- ☐ Week 7: Conceptual Questions 7-9

Friday, May 15
- ☐ Attend office hours
- ☐ Catch up or review the week's work

Statement of Academic Honesty

I affirm that the work completed from the packet is mine and that I completed it independently.

____
Student Signature

I affirm that, to the best of my knowledge, my child completed this work independently

____
Parent Signature

Monday, May 11

Happy Monday!

1. If technologically feasible, please send me a story from your life! I love reading them.
2. Check your answer to the practice problems from Week 6.
3. If you find it helpful, read or skim through my example responses for the conceptual questions from Week 6.
4. Complete all the practice problems in this week’s review (Week 7).

Tuesday, May 12

1. Answer, in full, complete, grammatical sentences, the first through fourth conceptual questions for Week 7. I will not be surprised if this takes you the whole of today’s 40 minutes. As it says in the instructions, you are writing as if you are teaching these concepts to someone whose only math instruction has been your answers to last week’s questions. (If you wish, you may answer these questions together, not in separate answers.)

Wednesday, May 13

1. Answer, in full, complete, grammatical sentences, the fifth and sixth conceptual questions for Week 7. I will not be surprised if this takes you the whole of today’s 40 minutes. As it says in the instructions, you are writing as if you are teaching these concepts to someone whose only math instruction has been your answers to last week’s questions. (If you wish, you may answer these questions together, not in separate answers.)

Thursday, May 14

1. Answer, in full, complete, grammatical sentences, the seventh through ninth conceptual questions for Week 7. I will not be surprised if this takes you the whole of today’s 40 minutes. As it says in the instructions, you are writing as if you are teaching these concepts to someone whose only math instruction has been your answers to last week’s questions. (If you wish, you may answer the first two of these questions together, not in separate answers. Question 9 really ought to be treated separately.)

Week 6 Practice Problems – Answer Key

Calculus I

Mr. Simmons

1. Given $f(x) = 3 - 5x - 2x^2$,

   (a)
   $$\begin{aligned} f(4) &= 3 - 5(4) - 2(4)^2 \\ &= 3 - 20 - 2 \cdot 16 \\ &= -17 - 32 \\ &= -49. \end{aligned}$$

   (b)
   $$\begin{aligned} f(0) &= 3 - 5(0) - 2(0)^2 \\ &= 3 - 0 - 0 \\ &= 3. \end{aligned}$$

   (c)
   $$\begin{aligned} f(-3) &= 3 - 5(-3) - 2(-3)^2 \\ &= 3 + 15 - 2 \cdot 9 \\ &= 18 - 18 \\ &= 0. \end{aligned}$$

   (d)
   $$\begin{aligned} & f(6-t) \\ &= 3 - 5(6-t) - 2(6-t)^2 \\ &= 3 - 30 + 5t - 2\left(36 - 12t + t^2\right) \\ &= -27 + 5t - 72 + 24t - 2t^2 \\ &= -2t^2 + 29t - 99. \end{aligned}$$

   (e)
   $$\begin{aligned} & f(7-4x) \\ &= 3 - 5(7-4x) - 2(7-4x)^2 \\ &= 3 - 35 + 20x - 2\left(29 - 56x + 16x^2\right) \\ &= -32 + 20x - 58 + 112x - 32x^2 \\ &= -32x^2 + 132x - 90. \end{aligned}$$

   (f)
   $$\begin{aligned} f(x+h) &= 3 - 5(x+h) - 2(x+h)^2 \\ &= 3 - 5x - 5h - 2\left(x^2 + 2xh + h^2\right) \\ &= 3 - 5x - 5h - 2x^2 - 4xh - 2h^2. \end{aligned}$$

2. Evaluate $\frac{f(x+h)-f(x)}{h}$ for

   (a) $f(x) = 4x - 9$.
   $$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{[4(x+h) - 9] - (4x - 9)}{h} \\ &= \frac{4x + 4h - 9 - 4x + 9}{h} \\ &= \frac{4h}{h} \\ &= 4 \quad (h \neq 0). \end{aligned}$$

   (b) $f(x) = \frac{2x}{3-x}$.
   $$\begin{aligned} & \frac{f(x+h) - f(x)}{h} \\ &= \frac{\frac{2(x+h)}{3-(x+h)} - \frac{2x}{3-x}}{h} \\ &= \frac{[2(x+h)](3-x) - (2x)[3-(x+h)]}{[3-(x+h)](3-x)} \cdot \frac{1}{h} \\ &= \frac{\left(-2x^2 + 6x - 2xh + 6h\right) - \left(-2x^2 + 6x - 2xh\right)}{[(3-x) - h](3-x)} \cdot \frac{1}{h} \\ &= \frac{6h}{(3-x)^2 - (3-x)h} \cdot \frac{1}{h} \\ &= \frac{6}{(3-x)^2 - (3-x)h} \quad (h \neq 0). \end{aligned}$$

3. Determine the domain of each function:
   (a) $f(x) = 3x^2 - 2x + 1$. The domain of f is the set of all real numbers.
   (b) $f(x) = -x^2 - 4x + 7$. The domain of f is the set of all real numbers.
   (c) $f(x) = 2 + \sqrt{x^2 + 1}$. The domain of f is the set of all real numbers such that $x^2 + 1 \geq 0$, which is all real numbers.
   (d) $f(x) = 5 - |x + 8|$. The domain of f is the set of all real numbers.

4. Find the following limits.

   (a)

$$\begin{aligned} &\lim_{x\to-5} \frac{x^2-25}{x^2+2x-15} \\ &= \lim_{x\to-5} \frac{(x+5)(x-5)}{(x+5)(x-3)} \\ &= \lim_{x\to-5} \frac{x-5}{x-3} \\ &= \frac{(-5)-5}{(-5)-3} \\ &= \frac{-10}{-8} \\ &= \frac{5}{4}. \end{aligned}$$

   (b)

$$\begin{aligned} \lim_{x\to1} \frac{x^2-1}{x+1} &= \frac{(1)^2-1}{(1)+1} \\ &= 0. \end{aligned}$$

   (c)

$$\begin{aligned} \lim_{x\to2} \frac{x^2-4}{x-2} &= \lim_{x\to2} \frac{(x-2)(x+2)}{x-2} \\ &= \lim_{x\to2} (x+2) \\ &= (2)+2 \\ &= 4. \end{aligned}$$

   (d)

$$\begin{aligned} \lim_{x\to3} \frac{x^2-4}{x-2} &= \frac{(3)^2-4}{(3)-2} \\ &= 5. \end{aligned}$$

   (e)

$$\begin{aligned} &\lim_{h\to0} \frac{(a+h)^2-a^2}{h} \\ &= \lim_{h\to0} \frac{a^2+2ah+h^2-a^2}{h} \\ &= \lim_{h\to0} \frac{2ah+h^2}{h} \\ &= \lim_{h\to0} (2a+h) \\ &= 2a+(0) \\ &= 2a. \end{aligned}$$

5. Let $f(x) = \begin{cases} 7-4x & \text{if } x<1 \\ x^2+2 & \text{if } x\geq1 \end{cases}$. Find the following limits:

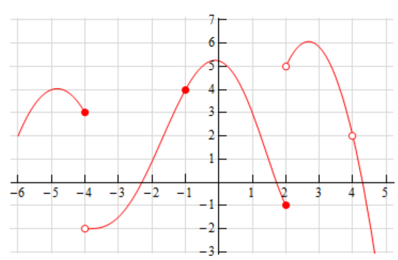
   (a)

$$\begin{aligned} \lim_{x\to-6} f(x) &= \lim_{x\to-6} (7-4x) \\ &= 7-4(-6) \\ &= 31. \end{aligned}$$

   (b)

$$\begin{aligned} \lim_{x\to1} f(x) &= \lim_{x\to1} \left(x^2+2\right) \\ &= (1)^2+2 \\ &= 3. \end{aligned}$$

6. In the function graph below, determine where the function is discontinuous.



A function is continuous at x if the limit at x equals the value at x. This function is discontinuous at -4 and 2 because the limit does not equal the value, and it is discontinuous at 4 because there it is undefined.

Week 6 Conceptual Questions – Example Responses

Calculus I
Mr. Simmons

There's not just one right answer these questions. Any answers that clearly conveys the concepts such that a non-math person could understand them is a good answer. Below are my example answers to each of the conceptual questions from Week 6.

1. In your own words, what is a function?

2. In your own words, what is a domain?

   Example. Functions are the main object of study in Calculus, but *function* isn't fundamentally a mathematical idea. The length of a person's shadow is a function of their height. When you're driving a car, your location is a function of time. The temperature outside is a function of variables in the weather, like whether clouds are covering up the sun. (*Variable* is also not fundamentally a mathematical idea. It just means something that can change, or vary.) One last example: the amount of money you have is a function of, among other things, the time you've spent earning it or spending it. In general, one thing being a function of another thing just means that the one is dependent on the other. Functions are relations of dependency.

   Where do functions fit into math? In pure math, we deal with abstract variables. Instead of talking about money as a function of time, we just talk about one variable as a function of another variable, and usually the variables are quantifiable, meaning we can assign them a number. And, so that we don't have to keep saying "this variable" or "that variable," we give them names. The names don't necessarily say anything about what the variables represent; they're just names... like proper names. The name "Simmons" doesn't tell you anything about me, really... except maybe that my ancestors were European. In math, if one variable is a function of another, we usually call the first one y and the second one x. (So, I guess the names x and y tell you that the ancestry of these variables is that they were named by a mathematician who likes conventional symbols.)

   One example of a function in pure math is the one represented by the equation

   $$y = 2x.$$

   (An equation is just a statement saying two things are actually just one and the same thing.) This equation says that y is actually just x taken two times. The relation between x and y here is a function. The variable y is dependent on x. It is a function of x. If x is 2, then y

has to be 4. If x is 10, then y has to be 20. We could call this the doubling function, because the dependent variable is double the independent variable.

We could even say that this function doubles things, as if the function is commiting an action. It takes inputs and doubles them to get outputs. The language of inputs and outputs is just another way of talking about functions. It makes them a bit more active, like they're doing something. But we're still just talking about a relation of dependency. The output is dependent on the input.

Finally, sometimes it's helpful to talk about what possible inputs a function can take. If temperature is a function of variables in the weather, what are those variables? (I'm not a meteorologist, so I sincerely don't know.) If money is a function of time, what are the possible inputs for the variable called "time"? Can it have negative numbers for inputs? Can it have fractions of numbers for inputs? We call the collection of all possible inputs to a function its "domain." *Domain* is also not fundamentally a mathematical concept. A king's domain is the land that he has power over. A function's domain is the same thing: it's the land that the function has power over. It's the collection of all things that the function could possible take as an input. You could say that a good king is dependent on his people; a function is dependent on its domain.

In math, we study specifically functions between abstract, variable quantities, but the fundamental idea is still the same as functions in everyday life: a function is a relation of dependency.

3. Exactly how are functions represented by equations?

   Example. If we have the equation $y = 2x$ and we plug in a number for x, there's only one possible number that y could be that could make the equation true. For example, if we plug in 2 for x, y would have to be 4. If we plug in 5 for x, y would have to be 10. For the doubling function to be represented by the equation $y = 2x$ means that it relates inputs that can be plugged in for x to the resulting out values that y would then have to be. So it relates 2 to 4 and 5 to 10, for the exact reasons that plugging 2 in for x yields 4 for y and plugging 5 in for x yields 10 for y, as described above. In general, for a equation to represent a function means that the function relates a given input value to a unique output value if and only if plugging the input value in for x in the equation makes it so that the one and only number that can be plugged in for y in the equation to make the equation true is the output value.

4. Exactly how are functions represented by graphs?

   Example. If a function relates real numbers to real numbers, it can be graphed on a cartesian coordinate plane. Draw a horizontal line and a vertical line and call them axes (singular *axis*). Their point of intersection is called the origin. Now every location on your piece of paper can be assigned a unique pair of numbers as a name based on its location relative to this origin. This unique pair of numbers comes simply from listing that location's horizontal distance from the origin next to its vertical distance from the origin, separated by a comma, all within parentheses. For example, the origin itself has the name $(0, 0)$, since it's zero units distant from itself horizontally and zero units distant from itself vertically. Draw a tick mark on the horizontal line exactly one unit to the right of the origin. (I say "unit" because we're not using this graph to represent any particular physical distance, but rather an abstract distance. If it makes it easier, replace every instance of the word *unit* in this paragraph with the word *inch*.) The point where that tick mark intersects the horizontal axis is called $(1, 0)$ because it is one unit to the right of the origin but not at all vertically distant from it. If you move exactly one unit up from there, you'd get the point $(1, 1)$. To distinguish the point that's one unit to the right of the origin from the one that's one unit to the left, we use negative numbers to mean leftward horizontal distance or downward vertical distance, so that the point one unit to the left of and one unit down from the origin will be called $(-1, -1)$.

That's a cartesian coordinate plane. But what does it have to do with functions? It seems we can represent a pair of numbers visually now, but is a function a pair of numbers?

Actually yes! Kind of. A function is a collection of pairs of numbers. We've described a function before now as taking inputs (maybe infinitely many inputs) and giving outputs, but each time that process happens, we can, as with the points on the plane, assign it a name, like $(0, 0)$. The doubling function would then have the pairs $(0, 0)$, $(1, 2)$, $(2, 4)$, $(3, 6)$, $(-2, -4)$, $(3.142, 6.283)$, and infinitely many more, since you can double any real number. If we fill in each of these points on the cartesian coordinate plane... voila! We've represented the doubling function. (Of course it's impossible to put your pencil down on paper and then pick it up again an infinite number of times, so we cheat by just drawing a line segment with arrows for endpoints... but there are some functions that actually cannot be drawn, because you really would have to put your pencil down and pick it up again an infinite number of times, for example the function whose inputs are real numbers and whose output for a rational input is 1 and whose output for an irrational numbers is 0.)

In general, if we consider a function to be a set of ordered pairs of the form (x, y), where x is an input and y is its unique output, then we can graph any function by filling in on the cartesian coordinate plane all the points whose names (which are pairs of numbers) are in the function.

[You might also mention and describe the vertical and horizontal line tests.]

5. In your own words, what is a limit?

Example. A limit is the tendency of a function. If distance driven is a function of time, then the tendency is for the distance to increase. As time passes, distance increases. Since the output is dependent on the input, limits will always state the tendency of an output as being dependent on the tendency of the input. For $y = 1/x$, as x gets bigger and bigger without bound, y gets closer and closer to 0. As x gets closer and closer to 0 from the positive side (perhaps with values like 1, 1/2, 1/4, 1/8, etc.), y gets bigger without bound. If x approaches 0 again but this time from the negative side of things (graphically the lefthand side of the origin), y gets bigger in the negative (downward) direction without bound. We would say that the limit of y as x approaches infinity is 0, the limit of y as x approaches 0 from the right is positive infinity, and the limit of y as x approaches 0 from the left is negative infinity. If $y = 1/x^2$, since y would approach positive infinity whether x is approaching 0 from the right or from the left, we say simply that y approaches infinity as x approaches 0 (without specifying from which direction). And limits don't have to include infinity: if $y = 2x$, then the limit of y as x approaches 5 is 10. These five statements can be stated symbolically as follows:

$$\lim_{x\to\infty}\frac{1}{x} = 0; \quad \lim_{x\to 0^+}\frac{1}{x} = \infty; \quad \lim_{x\to 0^-}\frac{1}{x} = -\infty; \quad \lim_{x\to 0}\frac{1}{x^2} = \infty; \quad \lim_{x\to 5} 2x = 10.$$

The first of these would be read, "The limit of $1/x$ as x approaches infinity is 0."

We've gotten by with minimal notation until now, but I think it will be helpful to introduce some function notation. If we name a function f, its output can be written as $f(x)$ ("f of x" or "the value of f at x"). So the function equations $y = 2x$, $y = 1/x$, and $y = 1/x^2$ (and many others) have the general equation $y = f(x)$.

So, informally, "$\lim_{x \to c} f(x) = L$" ("the limit of $f(x)$ as x approaches c is L" or "the limit of f near c is L") means that $f(x)$ can be made arbitrarily close to L by making x sufficiently close to, but not equal to, c. When dealing with infinity, simply replace "close to L" or "close to c" with "bigger or smaller without bound." Part of what's so powerful about limits is that phrase "but not equal to." If we had to make x equal to c, then we wouldn't be able to find the limit $\lim_{x \to 0} 1/x$, because $1/0$ is undefined.

To formalize concepts like "aribitrary" and "sufficient," mathematicians use the Greek letter ε ("epsilon") to denote an arbitrarily small quantity, and the Greek letter δ ("delta") to denote a sufficiently small quantity. Then, formally, "$\lim_{x \to c} f(x) = L$" means that given any arbitrarily small but still positive ε, we can find a sufficiently small positive δ such that all the x values within distance δ of c (but not c itself) will have resultant $f(x)$ values within distance ε of L. Still more concisely,

> "$\lim_{x \to c} f(x) = L$" means that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that, for all x, $0 < |x - c| < \delta$ implies $|f(x) - L| < \varepsilon$

($|x|$ being the "absolute value" function that outputs the positive version of the input, helpful for stating distances since they are always positive).

I would love to draw a diagram to make this concept more clear, Mr. Simmons, but we drew a million such diagrams in class and I'm typing on a computer so it would be really hard.

6. In your own words, what is a continuity?

   Example. Continuity is when the graph of a function is smooth. All the functions I've mentioned have been continuous, except the one I mentioned in parentheses as being undrawable. If you can graph a function without picking your pencil up, it's continuous, even if you have to use arrow symbols to indicate tendencies toward infinity (as with all the continuous functions we've mentioned).

   Another way to phrase it is that a continuous graph doesn't have any jumps or holes. We call those behaviors "discontinuities." Discontinuities happen when a function's behavior changes suddenly; that is, when its tendency, whose entire job is to tell you where you're going, fails at its job. When you're driving a car, the tendency is for distance driven to increase as time increases, and this tendency can be seen as a way of guessing where the car will be in the near future. If the car were to suddenly come to a halt (I'm sure you can imagine some reasons for this happening), the tendency of the time–distance function would have failed to predict what the function actually did. That would be an example of a jump discontinuity. A hole discontinuity might occur if you suddenly vanished from the earth, since then your position would be undefined.

   We defined a limit as a function's tendency. So a function is continuous if its limit, which tells you where you're going, is right. I see no need for further delay: formally, a function f is continuous at c if and only if

$$\lim_{x \to c} f(x) = f(c),$$

and a function is a continuous function if it is continuous at every point of its domain.

(A confusing note: the function defined by the equation $f(x) = 1/x$ is continuous, even though it is not continuous at $x = 0$, for the reason that 0 is not in its domain. You might say, or hear it said, that this function is discontinuous at 0, or that it has a discontinuity at 0. I suppose that's fine, since it's certainly not continuous at 0. But the only way to get a function to be not continuous is to have a jump discontinuity, a place in the domain where the function is defined but where the value at that point is not equal to the limit near that point.)

Continuous functions are nice because they are smooth, but they are not always as smooth as we'd like. Consider the absolute value function. The limit near 0 is 0 from both sides, yet the graph is not smooth there: there's a sort of "corner." We'll have to wait for derivatives to be able to identify the smoothest kinds of functions.

Week 7: Derivatives

Mr. Simmons
Calculus I

Practice Problems

Differentiate the following functions.

1. $f(x) = 6x^3 - 9x + 4$
2. $f(x) = 2x^4 - 10x^2 + 13x$
3. $f(x) = 4x^7 - 3x^{-7} + 9x$
4. $f(x) = \sqrt{x} + 8\sqrt[3]{x} - 2\sqrt[4]{x}$
5. $f(x) = \frac{4}{x} - \frac{1}{6x^3} + \frac{8}{x^5}$
6. $f(x) = \left(4x^2 - x\right)\left(x^3 - 8x^2 + 12\right)$
7. $f(x) = \frac{6x^2}{2-x}$
8. $f(x) = \left(6x^2 + 7x\right)^4$
9. $f(x) = \left(4x^2 - 3x + 2\right)^{-2}$
10. $f(x) = \sqrt[3]{1 - 8x}$
11. $f(x) = \frac{\sin(3x)}{1+x^2}$

Conceptual Questions

Answer the following questions in your own words. Try to avoid using symbols to the extent possible. Instead, write in full, complete, grammatical sentences. Answer these questions as if you're teaching these concepts to someone whose only math training has been to read and understand your answers to the conceptual questions from last week. That might mean giving examples, counterexamples, or analogies, for example. If you use any notation, it means explaining that notation, unless you explained it last week. (This is the most important part of the review.)

1. What is a slope?
2. What is a rate of change?
3. What is an average rate of change?
4. What is an instantaneous rate of change?
5. What is a derivative?
6. What is differentiation? (This question in particular you might be tempted to answer with only one sentence. That would be like answering "What is labor?" with "work; especially hard, physical work," without giving any examples like yardwork, carpentry, coal mining, factory work, etc. Be a teacher. Give examples. Name some known derivatives. I won't accept "finding a derivative" as a complete answer for a topic we spent multiple weeks on.)
7. What is an implicitly defined function?
8. What is implicit differentiation?
9. What is a differential?