GreatHearts
Irving

Remote Learning Packet

Please submit scans of written work in Google Classroom at the end of the week.

Week 8: May 18-22, 2020
Course: Calculus I
Teacher(s): Mr. Simmons

Weekly Plan:

Monday, May 18
☐ Story time!
☐ Review the significance of derivatives

Tuesday, May 19
☐ Practice problems

Wednesday, May 20
☐ Conceptual questions

Thursday, May 21
☐ Assessment

Friday, May 22
☐ Attend office hours
☐ Catch up or review the week's work

Statement of Academic Honesty

I affirm that the work completed from the packet is mine and that I completed it independently.

____
Student Signature

I affirm that, to the best of my knowledge, my child completed this work independently

____
Parent Signature

Monday, May 18

1. Happy Monday! If technologically feasible, please let me know how you're doing. Tell me a story from your life. Your summer plans. Your hopes and dreams. Your daily frustrations. Your hobbies. Looking forward to hearing from you!
2. Read the section entitled "Graphing Functions with Calculus Methods."

Tuesday, May 19

1. Complete the practice problems.

Wednesday, May 20

1. Look at the answers to the practice problems.
2. Answer, in full, complete, grammatical sentences, both conceptual questions for Week 8. As it says in the instructions, you are writing as if you are teaching these concepts to someone whose only math instruction has been your answers to last week's questions. (If you wish, you may answer these questions together, not in separate answers.)
3. Compare your answers to the conceptual questions with mine.

Thursday, May 21

1. On this day, I will post an assessment on Google Classroom. You will be able to complete it completely on the computer. The practice problems from this week and the past two weeks fully represent the kinds of questions that will be on this assessment. It will pull more heavily from this week's problems. Please email me if you have questions about these questions.

Week 8: Derivatives and Graphs

Mr. Simmons
Calculus I

Graphing Functions with Calculus Methods

Review each of the following theorems of calculus, and complete the practice problems that exemplify how the truths of the theorems help us graph and interpret graphs of functions more accurately than we could have with precalculus methods.

Theorem (Fermat's Theorem)**.** *If f is defined on (a, b) and has a local maximum (or minimum) at x, and f is differentiable at x, then $f'(x) = 0$.*

This is sometimes called Fermat's theorem of stationary points. It's the theorem we use to find the exact maximum or minimum values of polynomials.

But note that a derivative of $f'(x) = 0$ does not necessarily imply a maximum or minimum point at x. Consider $x = 0$ for $f(x) = x^3$. Then $f'(0) = 3(0)^2 = 0$, but $f(0) = (0)^3 = 0$ is certainly not a maximum nor minimum value of f. So how are we supposed to use this theorem to find maxima and minima? It's a first step. It finds us what we call critical points, and then we have to figure out whether a given critical point is a maximum or minimum.

The method for distinguishing between extreme values and other critical points builds up out of some corollaries to the Mean Value Theorem:

Theorem (Rolle's Theorem)**.** *If f is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, then there is a number x in (a, b) such that $f'(x) = 0$.*

This theorem states that if a differentiable function has the same value at two different points, it had to be flat at some point between the two. To phrase that in terms of motion, it means that if an object is in the same place at two different times, then the object had to be stationary at some time in between, even if only for a moment (as when a ball that's tossed directly upward is stationary for an instant at the apex of its trajectory).

Theorem (The Mean Value Theorem)**.** *If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a number x in (a, b) such that*

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

This theorem is a slanted version of Rolle's theorem. It states that at some point between any two points on a differentiable function, the exact rate of change had to equal the average rate of change. To phrase that in terms of motion, it means that if an object is, for example, at one point one moment and 60 miles away an hour later, than at some point between those two times it had to be going exactly 60 mph.s

Corollary (First). *If f is defined on an interval and $f'(x) = 0$ for all x in the interval, then f is constant on the interval.*

Corollary (Second). *If f and g are defined on the same interval, and $f'(x) = g'(x)$ for all x in the interval, then there is some number c such that $f = g + c$.*

Another way of phrasing this is to say that two functions with equal derivatives are either the same function or one of them is just the other shifted up or down.

Corollary (Third). *If $f'(x) > 0$ for all x in an interval, then f is increasing on the interval; if $f'(x) < 0$ for all x in the interval, then f is decreasing on the interval.*

Flowing from these corollaries is a method for finding extreme values exactly, sometimes called the "first derivative test." We use Fermat's theorem to find critical points, and then we use the first and third corollaries to the Mean Value Theorem to see if the critical points found are maxima, minima, or neither. Suppose that x is a critical point of f:

1. If $f' > 0$ in some interval to the left of x and $f' < 0$ in some interval to the right of x, then x is a local maximum point.
2. If $f' < 0$ in some interval to the left of x and $f' > 0$ in some interval to the right of x, then x is a local minimum point.
3. If f' has the same sign in some interval to the left of x as it has in some interval to the right, then x is neither a local maximum nor a local minimum point.

Once we have found what kind of critical point it is, we can calculate $f(x)$ to see what the exact value is.

We can now state a reliable method for graphing functions using the methods of calculus. To graph a function f, find

1. the critical points of f,
2. the value of f at the critical points,
3. the sign of f' in the regions between critical points (if this is not already clear),
4. the numbers x such that $f(x) = 0$ (if possible),
5. the behavior of $f(x)$ as x becomes large or large negative (if possible).

Although the location of local maxima and minima of a function is always revealed by a detailed sketch of its graph, it is usually unnecessary to do so much work. There is a popular test for local maxima and minima which depends on the behavior of the function only at its critical points.

Theorem. *Suppose $f'(a) = 0$. If $f''(a) > 0$, then f has a local maximum at a; if $f''(a) < 0$, then f has a local minimum at a.*

Theorem. *Suppose $f''(a)$ exists. If f has a local minimum at a, then $f''(a) \geq 0$; if f has a local maximum at a, then $f''(a) \leq 0$.*

The method that follows is sometimes called the "second derivative test." Suppose that x is a critical point of f and that f'' is continuous in a region around x:

1. If $f''(x) < 0$, then x is a local maximum point.
2. If $f''(x) > 0$ then x is a local minimum point.
3. If $f''(x) = 0$ then x can be a relative maximum, relative minimum, or neither.

Practice Problems

1. Show that there are no critical points for following functions.
   (a) $f(x) = \frac{1}{x}$
   (b) $f(x) = \frac{1}{x^2}$
   (c) $f(x) = \frac{3x+7}{x+2}$
   (d) $f(x) = \frac{x+3}{x^2-9}$
2. Graph the following functions using the methods of calculus:
   (a) $f(x) = x^4 - 2x^2$
   (b) $f(x) = \frac{x^2-2x+2}{x-1}$ (Hint: the slant asymptote of this graph is $y = x - 1$.)
3. Find the local maximum and local minimum (and say which is which) of the function defined by
$$f(x) = x^3 - x.$$
4. Consider the point $(3, 1)$ and the linear function $f(x) = (3/2)x$. Come up with a function d, where $d(x)$ is the distance between $(3, 1)$ and the point $(x, f(x))$. Minimize d; that is, find its minimum by finding which critical point is a minimum point and then evaluating d there. (Hint: d^2 and d have the same critical points, and it's easier to find them for d^2.) What does the minimum value of d that you just found represent?
5. Come up with a function A where $A(w)$ is the area of a rectangle of width w and perimeter P. (You'll need to find the height h in terms of w and P, noting perhaps that $2w + 2h = P$.) Find the value for w that maximizes $A(w)$.

Conceptual Questions

Answer the following questions in your own words. Try to avoid using symbols to the extent possible. Instead, write in full, complete, grammatical sentences. Answer these questions as if you're teaching these concepts to someone whose only math training has been to read and understand your answers to the conceptual questions from the last two weeks. That might mean giving examples, counterexamples, or analogies, for example. If you use any notation, it means explaining that notation, unless you explained it already. (This is the most important part of the review.)

1. How do derivatives help us graph functions more precisely than precalculus methods had allowed for?
2. How do derivatives help us maximimize or minimize (or optimize, meaning either maximize or minimize) functions?