GreatHearts Irving

Remote Learning Packet

There is no need to submit this packet at the end of the week. Enjoy your summer break!

Week 9: May 25-29, 2020
Course: Calculus I
Teacher(s): Mr. Simmons

Monday, May 25

Happy Memorial Day! No School!

Tuesday, May 26 - Friday May 29

For this week, we have a single challenging problem. There are three parts to the following document:

1. The problem.
2. An outline to a correct solution.
3. A correct solution.

I encourage everyone first to read only the problem and attempt to solve it. It is a particularly challenging problem, so once you've given it your best effort, if you're stuck, take a look at the outline and continue attempting the problem. If you find that you are still unable to solve it, read through the correct solution, and make sure you understand why this solution is correct. I also encourage you to come up with other interesting problems and try to solve them yourselves! Notice how this problem doesn't take long to state, but it is quite complex to solve.

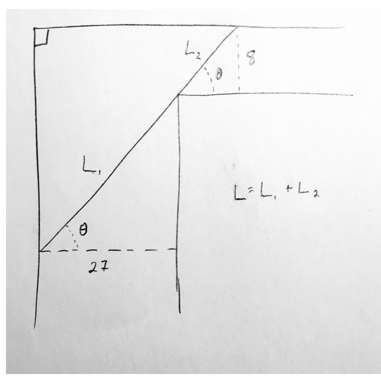
I hope that you have enjoyed this year. And next year, you'll be seniors. My, my. Enjoy your last summer of high school! If you have any fun summer plans or any parting words, feel free to email me!

A Challenging Calculus Problem

Calculus I
Mr. Simmons

Problem 1. A steel post is being carried down a hallway that is 2.7 meters wide. At the end of that hall, there is a right-angled turn into a narrower hallway that's only 0.8 meters wide. What is the length of the longest post that can be carried through this bend? This situation is represented below, with the slanted line representing the post. The diagram is a horizontal cross-section of the hallway at the exact level of the post. (I would suggest using decimeters instead of meters to avoid having to work with decimals.) You will need the following little-known identities from trigonometry:

$$\sin\left(\tan^{-1}\theta\right) = \frac{\theta}{\sqrt{1+\theta^2}} \quad \text{and} \quad \cos\left(\tan^{-1}\theta\right) = \frac{1}{\sqrt{1+\theta^2}}.$$



You have enough information to get started, but if you would like an outline of a correct solution, there is one on the next page.

Here is an oultine of a correct solution:

1. The diameter of the largest mirror that can be carried around the corner will be equal to the minimum of D.
2. Noting that $D = L_1 + L_2$, we find $D(\theta)$ by finding L_1 and L_2 in terms of θ. We do this using trigonometry, specifically the right-triangle definitions of sine and cosine (from SOH-CAH-TOA).
3. Having found an equation for $D(\theta)$, we find critical points of D by differentiating D, setting $D'(\theta) = 0$, and solving for θ.
4. Having found a critical point, which we know will be a minimum point, we evaluate $D(\theta)$ there to find its minimum value. This will be our final answer.

The complete solution starts on the next page.

Solution. We want to find the largest value for D, which represents the diameter of the mirror in the diagram. Looking at the diagram, we can see that this will be the minimum of D as a function of θ, where $D(\theta)$ represents the diameter of the largest mirror that can fit into the corner at the exact angle θ with the wall. Finding an equation for D in terms of L_1 and L_2 is trivial:

$$D = L_1 + L_2.$$

We can then put this in terms of θ by finding each of L_1 and L_2 in terms of θ, for which we use trigonometry. Note that

$$\cos\theta = \frac{27}{L_1} \quad \text{and that} \quad \sin = \frac{8}{L_2}.$$

Solving for L_1 and L_2, we get

$$L_1 = \frac{27}{\cos\theta} \quad \text{and} \quad L_2 = \frac{8}{\sin\theta}.$$

Then

$$\begin{aligned} D(\theta) &= L_1 + L_2 \\ &= \frac{27}{\cos\theta} + \frac{8}{\sin\theta}. \end{aligned}$$

Having found an equation for D, we can find its minimum by setting $D'(\theta) = 0$ and solving for θ:

$$\begin{aligned} D'(\theta) &= 0 \\ \frac{d}{d\theta}\left(\frac{27}{\cos\theta} + \frac{8}{\sin\theta}\right) &= 0 \\ \frac{d}{d\theta}\left(27(\cos\theta)^{-1} + 8(\sin\theta)^{-1}\right) &= 0 \\ -27(\cos\theta)^{-2}(-\sin\theta) - 8(\sin\theta)^{-2}(\cos\theta) &= 0 \\ \frac{27\sin\theta}{\cos^2\theta} - \frac{8\cos\theta}{\sin^2\theta} &= 0 \\ \frac{27\sin\theta}{\cos^2\theta} &= \frac{8\cos\theta}{\sin^2\theta} \\ 27\sin^3\theta &= 8\cos^3\theta \\ 3\sin\theta &= 2\cos\theta \\ \tan\theta &= \frac{2}{3} \\ \theta &= \tan^{-1}\left(\frac{2}{3}\right). \end{aligned}$$

We can even go through the trouble of verifying that this is a minimum point rather than some other type of critical point, but this is unnecessary, since we can tell by looking at the diagram this will be a minimum point. Now, to find out what the actual diameter is, we evaluate $D(\theta)$:

$$\begin{aligned}
D\left(\tan^{-1}\left(\frac{2}{3}\right)\right) &= \frac{27}{\cos\left(\tan^{-1}\left(\frac{2}{3}\right)\right)} + \frac{8}{\sin\left(\tan^{-1}\left(\frac{2}{3}\right)\right)} \\
&= \frac{27}{\left(\frac{1}{\sqrt{1+\left(\frac{2}{3}\right)^2}}\right)} + \frac{8}{\left(\frac{\left(\frac{2}{3}\right)}{\sqrt{1+\left(\frac{2}{3}\right)^2}}\right)} \\
&= 27\left(\sqrt{1+\frac{4}{9}}\right) + 8\left(\frac{\sqrt{1+\frac{4}{9}}}{\frac{2}{3}}\right) \\
&= 27\sqrt{\frac{13}{9}} + 8\left(\sqrt{\frac{13}{9}}\cdot\frac{3}{2}\right) \\
&= 9\sqrt{13} + 4\sqrt{13} \\
&= 13\sqrt{13}.
\end{aligned}$$

Therefore, the diameter of the largest mirror that can possibly be carried through the corner is $13\sqrt{13}$ decimeters, or approximately 4.687 meters.