GreatHearts Irving

Remote Learning Packet

There is no need to submit this packet at the end of the week. Enjoy your summer break!

Week 9: May 25-29, 2020
Course: 9 Geometry
Teacher(s): Mr. Mooney sean.mooney@greatheartsirving.org

Monday, May 25

Happy Memorial Day! No School!

Tuesday, May 26 - Friday May 29

Dear Students,
I hope you all are doing well! This week, there is just a single assignment, in two chapters. It is a little something I wrote, about sides and angles in triangles (trigonometry) and some other interesting ratios. It is intended to be *a combination of reading and discovery*. You may simply read through it all without stopping to really think on your own, and then it shouldn't take very long at all--but that wouldn't be any fun! There are a few challenging problems that I pose for you to solve, and I really, really encourage you, at those points, to stop reading, grab a pencil and paper, and try to work out the solutions for yourself. I won't be collecting any of the work that you do this week, so it does not have to be organized or neat or even legible to anyone but yourself! The point of the work that you do this week is, simply and purely, for your own enjoyment and intellectual growth.

And then, that is it! You will have completed your 9th grade study of Geometry.[1] You have come to know and understand the first half of Euclid's *Elements*, the standard of Geometry education for the last 2,500 or so years. This is no easy task! The depth that we went to in this class surpasses most high school and even college geometry courses these days. I am proud of all of you and how far you have come as mathematicians, scholars, and, in general, great-hearted young men and women. Your diligence in your studies and especially your eagerness to understand made it a true joy for me to teach you. I am very sad that we cannot end the year together, that I cannot tell you these things in person, but alas it is not possible. Do not be surprised if, at the beginning of next year, I come and interrupt your 10th grade math class to tell you how great you all are.

I wish you all a wonderful summer--full of recreation and true, good, and beautiful things--and I look forward to seeing you in the Fall.

Sincerely,
Mr. Mooney

[1] But only just started your life-long pursuit of it! In fact, I have suggestions for further study / summer reading if any of you would like it. Just email me!

Chapter One: Trigonometric Ratios

The Relationship of Angles and Sides in Triangles

You may have noticed in our study of Euclid's *Elements* a recurring question: what is the relationship of sides and angles in triangles? The question came up in many forms, and many propositions brought us a greater understanding. For example, we saw in I.5 and I.6 that equal angles and equal sides always go together in a single triangle, and then we saw in I.18 and I.19 that, if the angles are unequal, the greater side always subtends the greater angle.

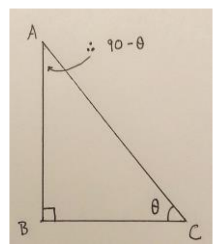
Now, having studied ratio and proportion, we might be wondering: how *exactly* do angles affect sides? By widening an angle by a certain amount, *how much longer* is the subtending side? Are sides and angles within a triangle proportional? Or is there some different kind of relation?

The truth is that angles and sides *are* related, and the answer does indeed lie in ratio and proportion, but not in the way that you might think. Let's take a look!

The Fundamental Theorem of Trigonometry

There is a special branch of Geometry that deals with this question, and it is called *Trigonometry*. The "-metry" part of the word comes from the Greek word meaning *measure*. The "Trigono-" part of the word comes from the Greek word (*trigonos)* meaning *triangle*.[1] So Trigonometry is simply "triangle measure"; and it deals with questions about the relationship of sides and angles in Triangles.

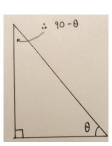
The beginning of Trigonometry is a simple realization, combining the truths of I.32 and VI.4. To see it, let's consider the right triangle ABC. Let's say that one of its acute angles has a measure of θ (pronounced "theta," a Greek letter often used to represent angle measures).

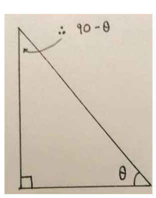


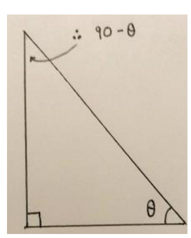
Now, since we know from I.32 that all triangles have angles that add up to two right angles, and since this triangle already has a right angle, we can know that the other two angles must add to one right angle, and therefore the remaining angle is $90-\theta$.

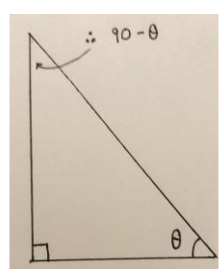
Now imagine another right triangle DEF, with a right angle and one acute angle with a measure of θ. Wouldn't it *also* have a remaining angle of $90-\theta$, by the same reasoning?

This means that every right triangle having an acute angle with a certain measure is equiangular and therefore similar (VI.4) to every other right triangle having an acute angle of that same measure. Or, put shortly: *every right triangle with an angle of θ is similar.*



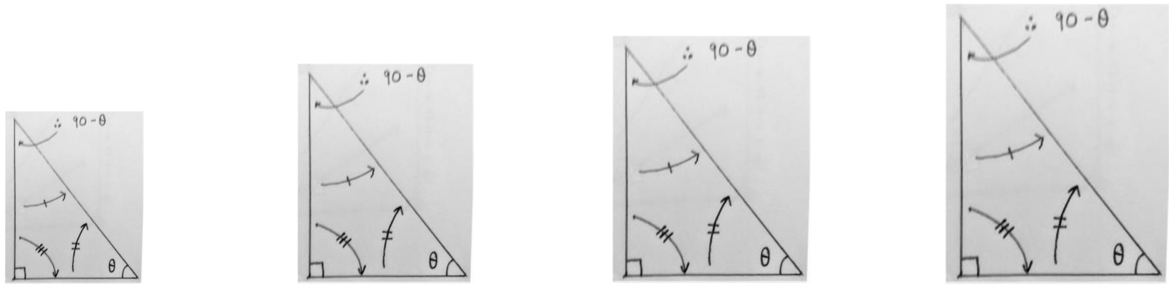






[1] Interesting to note that *trigonos* is actually a combination of *tri*, meaning three, and *gon*, meaning side. So the Greeks called it a "tri-side" while we say "tri-angle."

But similar triangles have sides that are in the same ratio: see the equal ratios in these triangles below:



This means that in every right triangle with an angle of θ, all the sides have constant ratios! Said another way:

In right triangles, the ratios of sides are uniquely determined by the measure of one acute angle.

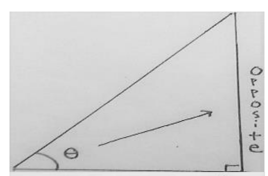
If you think of a right triangle and you say one of the angles is, for example, 30 degrees, then it will have ratios of sides that are the same as in *every* 30-degree right triangle. If we could figure out what those ratios are for our particular 30-degree triangle, we would know what they are for all of them.

Trigonometric Ratios: SOH CAH TOA

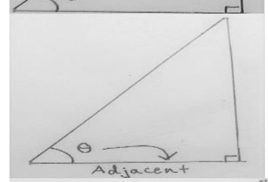
We will soon begin to investigate what the numerical values of these ratios are, but before we do, it is helpful to have some vocabulary for referring to specific ratios.

First, a way to <u>name the sides</u> of the triangle that you are talking about. From the perspective of the angle in question (θ) there are three names that we can give to the sides.

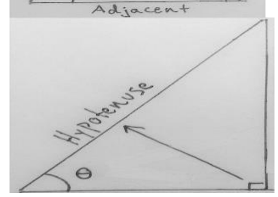
There is the side **opposite** the angle.



There is the side **adjacent** to the angle.



And then there is the **hypotenuse**. [2]



If we look at this right triangle, and we are thinking about how the measure of our angle would affect the ratios of the sides of the triangle, we see that there are three ratios to talk about.

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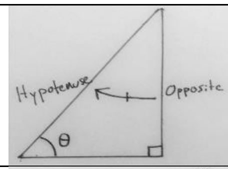
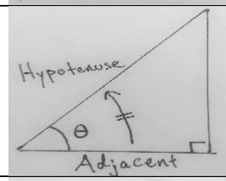
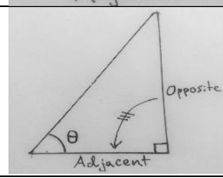
[2] Since the hypotenuse is always *opposite* the right angle, and we will only ever be thinking about ratios in terms of the acute angles, we will never get "opposite" and "hypotenuse" mixed up. Also, since the hypotenuse will always be *adjacent* to the acute angles, we name the side "adjacent" which is adjacent to the angle but is not the hypotenuse.

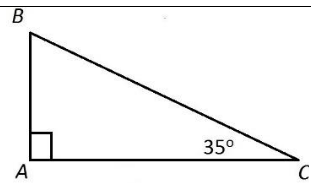
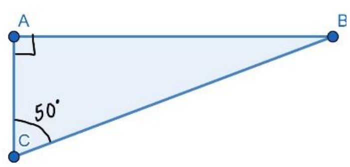
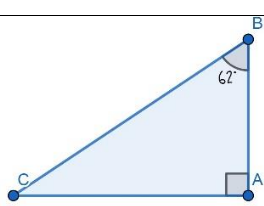
There is the ratio of the **opposite** to the **hypotenuse**.		We call this ratio the **Sine**.
There is the ratio of the **adjacent** to the **hypotenuse**.		We call this ratio the **Cosine**.
And then there is the ratio of the **opposite** to the **adjacent**.		We call this ratio the **Tangent.**

To remember these names in a convenient way, people remember "Soh Cah Toa." Try saying it out loud—it is rather fun to say. Soh Cah Toa! Soh Cah Toa! Soh Cah Toa!

Ok, maybe a little silly, but it's helpful! "SOH" stands for "**S**ine is **O**pposite: **H**ypotenuse." "CAH" stands for "**C**osine is **A**djacent : **H**ypotenuse." And "TOA" stands for "**T**angent is **O**pposite : **A**djacent."

We use these words—Sine, Cosine, and Tangent—in the following way. If we were wondering about the ratio of the opposite side to the hypotenuse when there is a 30-degree acute angle, we would say: "*What is the **sine** of 30 degrees*?"

This may take some getting used to. Please keep in mind that, even though we say "of 30 degrees," we are not exactly speaking about the 30-degree angle; we are talking about a specific *ratio of sides* that occurs when there is 30-degree angle in a triangle. Let's practice with this a bit.

In this triangle, what is the sine of 35 degrees? (Also written Sin50)		Sin35 = AB:AC
In this triangle, what is the cosine of 50 degrees? (Also written Cos50)		Cos50 = AC:BC
In this triangle, what is the tangent of 70 degrees? (Also written Tan70)		Tan62 = AC:AB

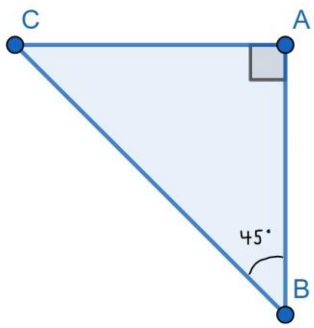
Getting the hang of it? Great! Now it's time for the real meat and potatoes of our inquiry.

Discovering Particular Trigonometric Ratios for Particular Angles

We said that the measure of an acute angle in a right triangle uniquely determines the ratios of sides. That is, since all right triangles with a 45-degree acute angle are similar, the **sine** of 45 degrees—the ratio of the opposite side to the hypotenuse—will *always be the same no matter which triangle it is*! If we were able to figure out what exactly that sine ratio is in one triangle (is it a 1:2 ratio? a 3:4 ratio? a 5:3 ratio? etc.) then we would know what the ratio is *in every 45-degree-angle triangle*.

And that is, indeed, the question that we will begin with. *What are the sine, cosine, and tangent ratios in a right triangle with respect to a 45-degree angle?*

We are looking for a particular numerical value for the ratio: as we noted above, something like 1:2, or 5:3. But looking at the diagram of ABC to the right, we might wonder how it is possible to get any numbers out of it. We don't know the length of any of them!



The trick here is that, since we know that all 45-degree right triangles are similar, *we can choose a length to get us started and it does not matter what length we choose*. We cannot choose *all* of the lengths, because then we would be determining the shape of the triangle in a way not based on the angle, but if we choose one side length and figure out the others based on it, we will not distort the shape.

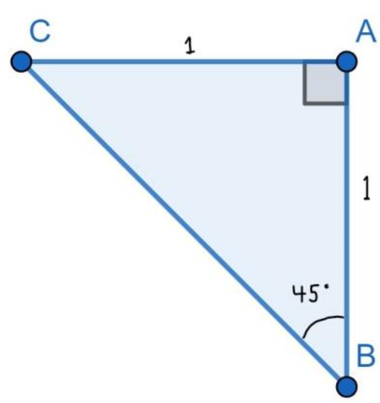
So let's say that the length of AB is 1. Now, *can we find the length of the other sides?* Now if the angle we are looking at is 45 degrees, what is the measure of the other angle? Knowing that they all must add up to 180 degrees, we can see that the remaining angle is also 45 degrees, making this an isosceles triangle. Therefore, if we say AB is 1, then BC also equals 1.

And already we are able to answer one of our questions: what is the **tangent** of 45 degrees? Remember that tangent is the ratio of the side opposite our angle to the side adjacent to our angle. Here, we have a 1:1 ratio of sides. We can therefore say that Tan45 = 1. (And do you see that, had you chosen 3 as your length for AB, then BC would also have to be 3, and you would get a 3:3 ratio, which is equivalent to a 1:1 ratio. $\frac{1}{1} = \frac{3}{3} = 1$. The same, of course, would be true for any number you originally chose.)

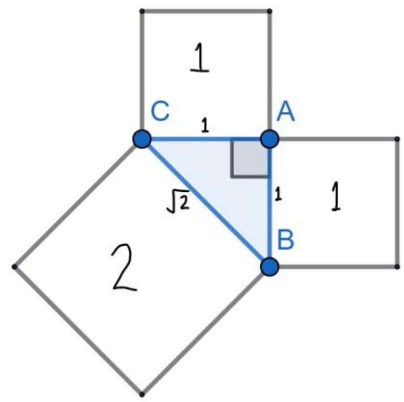
Thus we have the tangent of 45 degrees! This is a great success! 45-degree angles in right triangles will always result in a 1:1 ratio of opposite and adjacent sides. But how to determine the sine and cosine? These both involve the hypotenuse, whose length we do not know yet.

I don't want to give away all of the fun of discovery, so let me pause here and let you try to discover it on your own.

Try to discover the length of the hypotenuse, and then use that to discover the remaining ratios: sin45 and cos45. The solution will follow on the next page but, to get you started, let me give you a little hint: think about that very important and glorious Book I proposition that tells us about the relation of sides of a right triangle!



How did it go? Did you find them? If you discovered them on your own, you noticed that the way to determine the length of the hypotenuse was with the Pythagorean theorem (I.47). If each of the legs have lengths of 1, then the squares on them have areas of 1. But the area of the square on the hypotenuse equals the sum of the areas of the squares on the legs. Therefore, the area of the square on the hypotenuse is equal to 2.



To determine the length of the side of the square, we just need to ask *what number times itself makes two?* This should ring a bell from last year: the answer to that question is the square root of two (written $\sqrt{2}$).

Therefore, the sine of 45 degrees is the ratio 1: $\sqrt{2}$. And the cosine of 45 is also 1: $\sqrt{2}$.

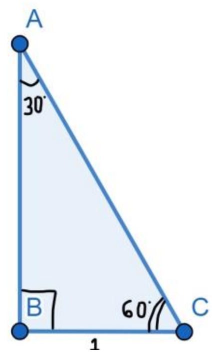
Thus, $\sin 45 = \frac{1}{\sqrt{2}}$, $\cos 45 = \frac{1}{\sqrt{2}}$, and $\tan 45 = 1$. The sine and cosine can be further simplified or turned into decimals, but we will leave them as they are for now.[3]

Congratulations, we have discovered our very first trig ratios! These values are universal, and are the same for *every 45-degree right triangle!*

Let's try to discover some more! You'll notice that the 45-degree angle was a nice choice because it resulted in an isosceles triangle. Not all angles work out so nicely. Here is one that, while trickier than 45 degrees, also has a solution that, with a little creativity, we can come to on our own.

Find the sine, cosine, and tangent of 30 degrees. Then, while you're at it, *find the sine, cosine, and tangent of 60 degrees*. (I say "while you're at it," because when one angle in a right triangle is 30 degrees, the other angle is 60 degrees.)

To help you get started on this, let's choose the length of one side. Let's make BC = 1. If you would like a very helpful hint, see the footnote at the bottom of this page.[4] The solution will follow on the next page.

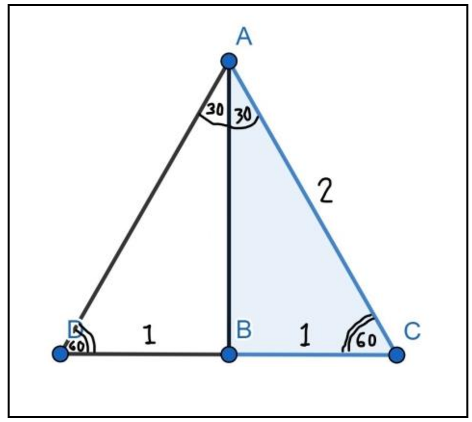


[3] Those of you familiar with the "no roots in the denominator" rule and the trick for simplifying them will know that it is more proper to write the sine and cosine ratios as $\frac{\sqrt{2}}{2}$.

[4] Since 60 degrees is the angle measure of an equilateral triangle, try turning your triangle into the half of a larger equilateral triangle. That may help you find a second side length. (And if you find a second side length of a right triangle, you know how to find the third!)

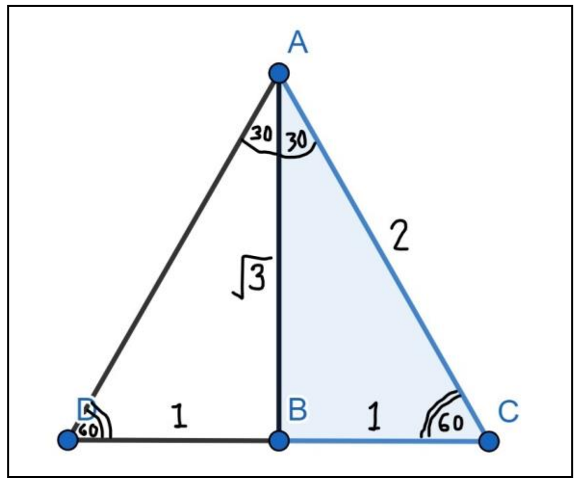
How did it go? We will go through the solution in steps, so if you have not gotten the solution yet, and you read something helpful here, I encourage you to stop, go back to the previous page, and see if you can get it from there.

The first trick is, as I said in the footnote hint, to make your triangle the one half of a larger equilateral triangle, by creating a congruent triangle on the other side of AB:



Therefore, since all the angles would be 60 degrees each, the triangle is equilateral. Furthermore, since each half would be congruent, if BC = 1, then the whole DC would have to equal 2. That in turn, since the triangle is equilateral, would mean that AC is also equal to 2. And we have a second side length!

From there, we can reason by I.47 to the length of the third side of our original right triangle.



The square on 2 would have an area of four, and the square on 1 would have an area of 1. Therefore, the square on the other leg would have to have an area of 3. Therefore, the length of the remaining side must be the square root of three: $\sqrt{3}$.

Therefore, the trigonometric ratios are as follows:

$sin30 = \frac{1}{2}$ $cos30 = \frac{\sqrt{3}}{2}$ $tan30 = \frac{1}{\sqrt{3}}$

$sin60 = \frac{\sqrt{3}}{2}$ $cos60 = \frac{1}{2}$ $tan60 = \frac{\sqrt{3}}{1}$

Congratulations! You have discovered the trigonometric ratios for a total of three different angle measures! This is no small accomplishment.

Though we won't do this ourselves, mathematicians have found ways of knowing the ratios for any acute angle measure, and have organized them into a chart called a "Table of Trigonometric Values," as you can see for yourself below. You will see that there is a numerical value for the sine, cosine, and tangent ratios for every whole-number angle between 0 and 90 degrees.[5]

[5] What would it mean to have a trigonometric ratio of 0 or 90 degrees? Is that possible?

Trigonometry Table

A	SIN(A)	COS(A)	Tan(A)
0	0.0000	1.0000	0.0000
1	0.0175	0.9998	0.0175
2	0.0349	0.9994	0.0349
3	0.0523	0.9986	0.0524
4	0.0698	0.9976	0.0699
5	0.0872	0.9962	0.0875
6	0.1045	0.9945	0.1051
7	0.1219	0.9925	0.1228
8	0.1392	0.9903	0.1405
9	0.1564	0.9877	0.1584
10	0.1736	0.9848	0.1763
11	0.1908	0.9816	0.1944
12	0.2079	0.9781	0.2126
13	0.2250	0.9744	0.2309
14	0.2419	0.9703	0.2493
15	0.2588	0.9659	0.2679
16	0.2756	0.9613	0.2867
17	0.2924	0.9563	0.3057
18	0.3090	0.9511	0.3249
19	0.3256	0.9455	0.3443
20	0.3420	0.9397	0.3640
21	0.3584	0.9336	0.3839
22	0.3746	0.9272	0.4040
23	0.3907	0.9205	0.4245
24	0.4067	0.9135	0.4452
25	0.4226	0.9063	0.4663
26	0.4384	0.8988	0.4877
27	0.4540	0.8910	0.5095
28	0.4695	0.8829	0.5317
29	0.4848	0.8746	0.5543
30	0.5000	0.8660	0.5774
31	0.5150	0.8572	0.6009
32	0.5299	0.8480	0.6249
33	0.5446	0.8387	0.6494
34	0.5592	0.8290	0.6745
35	0.5736	0.8192	0.7002
36	0.5878	0.8090	0.7265
37	0.6018	0.7986	0.7536
38	0.6157	0.7880	0.7813
39	0.6293	0.7771	0.8098
40	0.6428	0.7660	0.8391
41	0.6561	0.7547	0.8693
42	0.6691	0.7431	0.9004
43	0.6820	0.7314	0.9325
44	0.6947	0.7193	0.9657
45	0.7071	0.7071	1.0000

A	SIN(A)	COS(A)	Tan(A)
45	0.7071	0.7071	1.0000
46	0.7193	0.6947	1.0355
47	0.7314	0.6820	1.0724
48	0.7431	0.6691	1.1106
49	0.7547	0.6561	1.1504
50	0.7660	0.6428	1.1918
51	0.7771	0.6293	1.2349
52	0.7880	0.6157	1.2799
53	0.7986	0.6018	1.3270
54	0.8090	0.5878	1.3764
55	0.8192	0.5736	1.4281
56	0.8290	0.5592	1.4826
57	0.8387	0.5446	1.5399
58	0.8480	0.5299	1.6003
59	0.8572	0.5150	1.6643
60	0.8660	0.5000	1.7321
61	0.8746	0.4848	1.8040
62	0.8829	0.4695	1.8807
63	0.8910	0.4540	1.9626
64	0.8988	0.4384	2.0503
65	0.9063	0.4226	2.1445
66	0.9135	0.4067	2.2460
67	0.9205	0.3907	2.3559
68	0.9272	0.3746	2.4751
69	0.9336	0.3584	2.6051
70	0.9397	0.3420	2.7475
71	0.9455	0.3256	2.9042
72	0.9511	0.3090	3.0777
73	0.9563	0.2924	3.2709
74	0.9613	0.2756	3.4874
75	0.9659	0.2588	3.7321
76	0.9703	0.2419	4.0108
77	0.9744	0.2250	4.3315
78	0.9781	0.2079	4.7046
79	0.9816	0.1908	5.1446
80	0.9848	0.1736	5.6713
81	0.9877	0.1564	6.3138
82	0.9903	0.1392	7.1154
83	0.9925	0.1219	8.1443
84	0.9945	0.1045	9.5144
85	0.9962	0.0872	11.4301
86	0.9976	0.0698	14.3007
87	0.9986	0.0523	19.0811
88	0.9994	0.0349	28.6363
89	0.9998	0.0175	57.2900
90	1.0000	0.0000	∞

Chapter Two: Other Important and Interesting Ratios

The Golden Ratio

If you have followed along up until now, nice work! If you spent time trying to figure out each ratio on your own, it could have taken you a long time. If you are not too tired or worn out, I have one last problem I want to share with you.

Earlier this quarter, you learned about the legendary "Golden Ratio." I mentioned to you at the time that it was difficult to understand it fully without a solid foundation in ratio and proportion. We now have that foundation, so let us return.

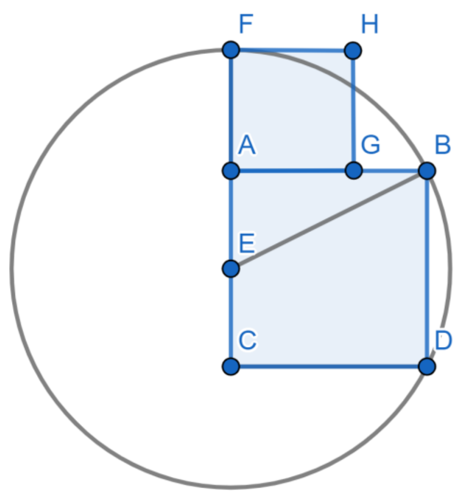
The Golden Ratio is a ratio of the lengths in a line cut at a particular point, such that the whole is to the larger part as the larger part is to the smaller part. Picture a line cut into parts *a* and *b*, such that $(a+b):a :: a:b$.

Each of these ratios, $(a+b):a$ and $a:b$, are the Golden Ratio. As you can see, for any given line, there is only one place to cut it so that this proportion emerges.   We used II.11 to find that very point.

Can you guess what's coming? If the Golden Ratio is always the same, can we find a specific number for it? For example, is it 2:1? Or 4:3? *What is the numerical representation of the Golden Ratio?*

We can easily see by a kind of guess-and-check method a lot of ratios that it is *not*. For example, 2:1 does not work because the proportion $(2+1):2 :: 2:1$ is clearly false. We see the same result for 4:3 and many other ratios we might be tempted to try.

Instead of guess-and-check, let's try a method similar to the one we used earlier. Let's look at a diagram—our construction of the Golden Ratio. Here, AB was the original line, and we cut it in the Golden Ratio at G, such that $AB:AG :: AG:GB$.



If we let AB = 1, find the ratio of AB:AG or AG:GB and you will know the numerical value of the Golden Ratio!

Just to refresh you on the construction,

1) square ABCD was built on AB
2) AC was bisected at E
3) Circle FBD was drawn with center E and radius EB
4) Square AFGH was built on AF.

Now, letting AB=1, find AB:AG.[6]   Here is my recommendation about the order in which to proceed:

1) Find AC and then AE
2) Find EB
3) Find AF
4) Find AG
5) Write AB:AG as a ratio. This will be the Golden Ratio!

Give yourself some time to discover the ratio on your own. The solution is on the following page.

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[6] I recommend finding AB:AG, rather than AG:GB, just to avoid some difficulty simplifying fractions later on.

Here is the solution:

1) Since AB = 1, AC must = 1 also, because ABCD is a square. Since AC was bisected, AE = $\frac{1}{2}$
2) Since AE= ½, and AB = 1, therefore EB = $\sqrt{\frac{5}{4}}$, which simplified is equal to $\frac{\sqrt{5}}{2}$
3) Since EB = $\frac{\sqrt{5}}{2}$, EF = $\frac{\sqrt{5}}{2}$ as well, because they are radii of the same circle FBD.
4) Since EF = $\frac{\sqrt{5}}{2}$, and AE = ½ , AF equals the difference between them: $\frac{\sqrt{5}}{2} - \frac{1}{2}$, or $\frac{\sqrt{5}-1}{2}$
5) Since AF = $\frac{\sqrt{5}-1}{2}$, and AF and AG are sides of the same square, AG = $\frac{\sqrt{5}-1}{2}$ as well.
6) Since AB = 1, and AG = $\frac{\sqrt{5}-1}{2}$, the Golden Ratio is $\frac{1}{\frac{\sqrt{5}-1}{2}}$.
7) Simplifying this,[7] we get $\frac{1+\sqrt{5}}{2}$. This is the numerical value of the Golden Ratio!

The Golden Ratio, thus, is an irrational number. The decimal approximation is ≈ 1.618033989. The symbol traditionally used to represent this number is φ (the Greek letter "phi").

The Fibonacci Series

The Golden Ratio has a very interesting connection to the Fibonacci Series. Essentially, the Fibonacci Series is a series of numbers, beginning with 0 and 1, and following a pattern in which the next term in the series is arrived at by adding the previous two terms. E.g. to get the 3rd term of the series, add 0 and 1.

$$\mathbf{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144} \ldots$$

The Fibonacci Series, oddly enough, was first discovered as a model to represent reproduction patterns of rabbits, and then was subsequently discovered in more and more natural phenomena. For example, pinecones and nautilus shells follow a Fibonacci pattern of growth. Very mysterious!

Now, try taking any two numbers in the sequence, and take the latter in ratio to the former. For example, $5:3$, or $8:5$, or $55:34$. It turns out that, as you go further along in the pattern (moving out further and further to the right), the ratios get closer and closer to φ (the Golden Ratio). Try it yourself! If you divide any pair of adjacent numbers, the latter by the former, the number will get closer and closer to our decimal approximation: 1.618033989.

A mathematician eventually *proved* that, as the series goes to infinity, the ratio becomes equal to φ.

The Ratio of Circumference to Diameter

Another very important ratio is the ratio of the circumference to the diameter in a circle. It was noticed thousands of years ago that a circle's circumference was always about three times longer than its diameter: that is, the ratio of circumference to diameter was about 3:1. It has since been discovered that the ratio is irrational, and it is represented with the Greek letter "pi": that is, π. A decimal approximation of this ratio is ≈ 3.141592654. (That is, the circumference of every circle is roughly 3.14 times longer than its diameter. Thus, $C = \pi d$). How could you get a numerical value for π…? It's a great question but, unfortunately, we are out of time in this packet. Maybe another time!

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[7] The simplification of this involves some tricky algebra. Multiply by $\frac{\frac{2}{\sqrt{5}-1}}{\frac{2}{\sqrt{5}-1}}$, and then by $\frac{1+\sqrt{5}}{1+\sqrt{5}}$ in order to eliminate the resulting radical in the denominator.

Congratulations! You are now officially done! Thank you again for your hard work this year. And now, as a little parting gift to you all, here is some ancient wisdom about the beauty and power of mathematics.

"[Mathematics is] the bridge that takes us from the senses and opinions to the mind and understanding, from the concrete and familiar objects to immaterial and eternal abstractions, from matter to soul."

- Nichomachus of Gerasa

Then, my noble friend, geometry will draw the soul towards truth, and create the spirit of philosophy.

- Plato

"The knowledge of which geometry aims is the knowledge of the eternal."

- Plato

"Those who assert that the mathematical sciences say nothing of the beautiful or the good are in error."

- Aristotle